77 Fe

ELEMENTS
OF
Plane Geometry.

To which are added,

An Essay on the MAXIMA and
MINIMA of Geometrical Quantities,

AND

A brief Treatiſe of regular SOLIDS;

ALSO,

The MENSURATION of both Superficies
and Solids, together with the Construction of a
large Variety of Geometrical PROBLEMS.

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Deſigned for the Use of Schools.

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BRITAN
NICVM

PREFACE.

MY Deſign, in this little Book, is to lay, before the young Beginner, an eaſy Method for acquiring a competent Knowledge in the Subject of Geometry, *without, either being obliged to go through a Number of uſeleſs and tedious Propoſitions, or having recourſe to the* ungeometrical *Methods of Demonſtration, which abound in moſt modern Compoſitions of this Nature.*

I am ſenſible, indeed, of the Difficulty of the Undertaking, and doubt not but ſome may, even, *look upon it as a Piece of Preſumption, in me, to hope to ſucceed after ſo great and judicious an Author as* Euclid: *However, as I have no Deſign to depreciate the Value of a Work which has been univerſally eſteemed, and ſtood the Teſt of Ages, neither ſhall I ſcruple to affirm that the Reader will alſo find ſome Things in the following Sheets not altogether unworthy the Peruſal, which are no where elſe to be met with.*

And, as the Pleaſure and Satisfaction ariſing to the Mind from the Study of Geometry, *chiefly, conſiſts in the*

the Accuracy of its Reaſoning and the Certainty of its Concluſions, I have all along, in Imitation of that excellent Author, endeavoured to keep up that Spirit and Rigour of Demonſtration which is ſo neceſſary, and, in a manner, peculiar to the Subject.

In order to this it ſeem'd convenient to add a new Poſtulatum to thoſe before laid down by Euclid; *but, then, what is there required, I believe, no one will deny; and, were it not a Crime, I could make it appear, that even* Euclid *himſelf has taken ſome Things for granted, not leſs conſiderable, and that too in the Body of his Work. As, in* Prop. 22. Book I. *where he tells you, that Lines drawn from two given Points to the Interſection of two Circles, will anſwer the Conditions of the Problem, without, firſt, proving thoſe Circles muſt neceſſarily interſect each other: I could point out ſome other Particulars of the like Nature; which I would not be thought to mention as Faults, but rather by way of Apology for any little Slips or Overſights I may be found guilty of; by ſhewing how impoſſible it is to expect a Work quite perfect.*

As to the Doctrine of Proportions, about which there has been ſo much Controverſy, I have in this, ſtill, differed more from Euclid. *No one will, I believe, deny the Accuracy of his Method, nor can be ignorant of the Motives that induced him to it; namely, the Buſineſs of Incommenſurables: But that his Definitions of Ratio and Proportion are neither ſo natural nor ſo eaſy as might be wiſh'd for, the many Diſputes about them, among the Learned, will in a good Meaſure confirm.*

The proper Deſign of a Definition being to deſcribe and explain the Term or Thing defined, ſo as to give a preciſe and adequate Idea thereof, it is certain that the exact-meaning of every Term made uſe of in a Definition ought to be perfectly underſtood, or, at leaſt, ſhould be *better known and more commonly received than the Term to be defined.*

An Author may indeed, to guard againſt Objections, take Shelter in Obſcurity, by uſing Terms of a lax and indeterminate Signification, or ſuch as want themſelves to be defined; but his Definition, however labour'd, can never be called a good one if it wants Perſpicuity. It is upon this Conſideration that I have not at all attempted to give a Definition of the Term Proportion; ſince no other Words appear to me ſo ſignificant and expreſſive, of the Idea annexed thereto, as the Term itſelf in the common Acceptation of it.

It was my Intent, at firſt, to have ended this Book with the Elements, *without touching at all upon* Solids, *but being perſuaded that the Addition of ſomething upon this Head, would render the Work more univerſally uſeful, I have, in the Appendix, alſo laid down the moſt conſiderable Properties and Proportions of ſolid Bodies; which, tho' put down and demonſtrated in a Manner ſomewhat more looſe than the Elements, for the ſake of Eaſe and Brevity, will, I hope, be found ſufficiently clear and evident. As to what follows afterwards upon* Menſuration, *it is very ſhort, and will be of uſe, at leaſt, to thoſe who have not ſeen other Books on the Subject.— But as to the* Maxima *and* Minima *and the geometrical Conſtructions, theſe, I flatter myſelf cannot fail of pleaſing ſome, and that even thoſe, who have already made conſiderable Advances in the Study of* Geometry, *may meet with ſome Things there, not unworthy their Approbation.*

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ELEMENTS

OF

Plane Geometry.

BOOK I.

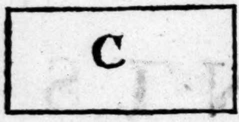
DEFINITIONS.

1. GEOMETRY is that Science, by which we compare ſuch Quantities together as have Extenſion.

Extenſion is diſtinguiſhed into Length, Breadth and Thickneſs.

2. A Line is that which has Length without Breadth, as AB. A ——— B

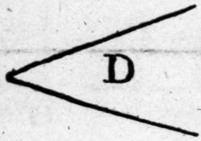
The Terms, Bounds, or Extremes of a Line are call'd Points, and have no Magnitude or Extenſion.



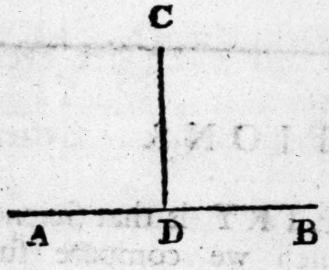
3. A Surface is that which has Length and Breadth only, as C.

The Bounds of a Surface are Lines.

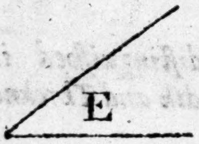
4. A Right-line is that which lies evenly betwixt its Extremes, or which every where tends the ſame way, as A B. A———B



5. An Angle is the Inclination, or Opening of two Right-lines meeting each other in a Point, as D.

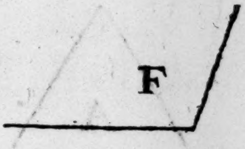


6. When one Right-line C D, falling upon another A B, makes the Angles on both Sides equal, thoſe Angles are called Right-angles; and the Line C D is ſaid to be perpendicular to the Line A B.



7. An acute Angle is that which is leſs than a Right-angle, as E.

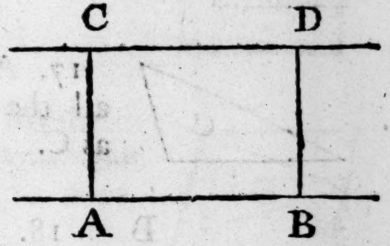
8. An obtuſe Angle is that which is greater than a Right-angle, as F.



9. The Diſtance of two Points is the right Line reaching from one Point to the other.

10. The Diſtance of a Point from a Line, is its Diſtance from the neareſt Point in that Line.

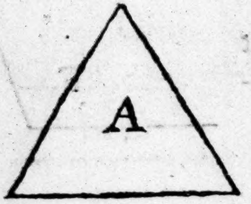
11. Two Right-lines AB, CD are ſaid to be parallel, or equidiſtant, when Perpendiculars AC and BD, to one of them AB, any where taken, terminated by the other CD, are equal the one to the other.



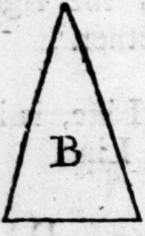
12. A plane Figure is that which lies evenly betwixt its Bounds or Extremes, or which agrees with any Right-line reaching from one Extreme to another.

13. A right-lined plane Figure is that whoſe Terms or Bounds are Right-lines.

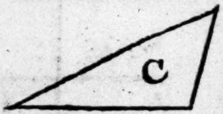
14. All plane Figures bounded by three Right-lines are called Triangles.



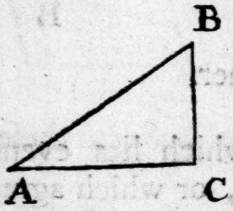
15. An equilateral Triangle is that whoſe Bounds or Sides are all equal, as A.



16. An iſoſceles Triangle is when only two Sides are equal, as B.



17. A ſcalene Triangle is when all the three Sides are unequal, as C.



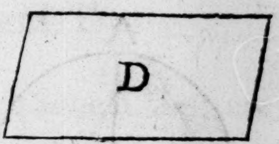
18. A right-angled Triangle is that which has one Right-angle, as ABC; whereof the Side AB oppoſite to the Right-angle is called the Hypothenuſe.

19. An obtuſe-angled Triangle is that which has one obtuſe Angle.

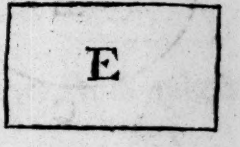
20. An acute-angled Triangle is that which has all its Angles acute.

21. Every plane Figure bounded by four Right-lines is called a Quadrilateral.

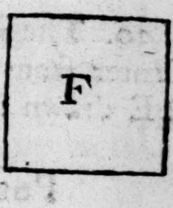
22. Any Quadrilateral whoſe oppoſite Sides are parallel is called a Parallelogram, as D.



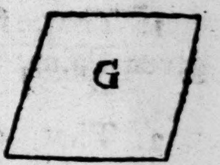
23. A Parallelogram whoſe Angles are Right, is called a Rectangle, as E.



24. A Square is a Rectangle whoſe four Sides (as well as Angles) are all equal, as F.



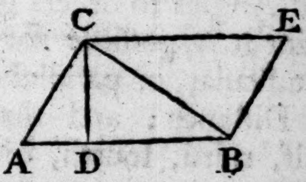
25. A Rhombus is a Parallelogram whoſe Sides are all equal but its Angles not Right, as G.

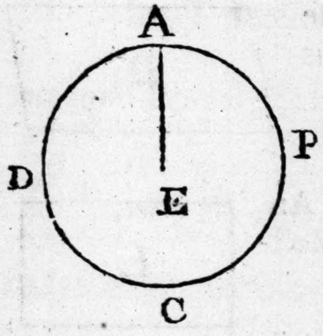


26. All other four-ſided Figures beſides theſe are called Trapeziums.

27. A Right-line joining any two oppoſite Angles of a four-ſided Figure, is called a Diagonal.

28. That Side A B upon which any Parallelogram A C E B, or Triangle A C B is ſuppoſed to ſtand is called the Baſe; and the Perpendicular C D falling thereon from the oppoſite Angle C, is called the Altitude of the Parallelogram or Triangle.





29. A Circle is a plane Figure bounded by one continued Line APCD, called its Circumference, every where equally diſtant from a Point E within the Circle, called its Center.

30. The Radius of a Circle is the Diſtance of the Center from the Circumference, or a Right-line AE drawn from the Center to the Circumference.

Poſtulates *or* Petitions.

1. That, from any given Point to any other given Point, a Right-line may be drawn.

2. That, a Right-line may be produced, or continued out at Pleaſure.

3. That from any Center, at any Diſtance, or with any Radius, the Circumference of a Circle may be deſcribed.

4. It is alſo required, that the Equality of Lines and Angles to others given, be granted as poſſible: That it is poſſible for one Right-line to be perpendicular, or parallel to another, at a given Point, or Diſtance; and that every Magnitude has its half, third, fourth, *&c.* Part.

Note, Though theſe Poſtulates are not always quoted, the Reader will eaſily perceive where, and in what Senſe they are to be underſtood.

AXIOMS, or Self-evident Truths.

1. Things, equal to one and the ſame Thing, are alſo equal to each other.

2. Every Whole is greater than its Part.

3. Every Whole is equal to all its Parts taken together.

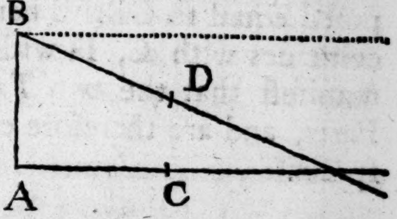
4. If to equal Things, equal Things be added, the Wholes will be equal.

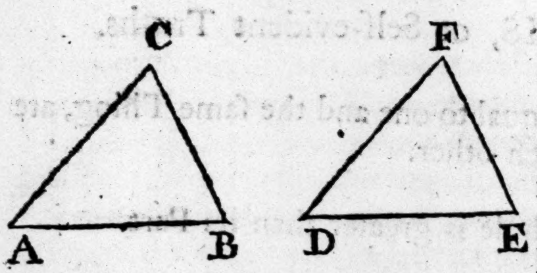
5. If from equal Things, equal Things be taken away, the Remainders will be equal.

6. If to, or from unequal Things, equal Things be added, or taken away, the Sums or Differences will be unequal.

7. All Right-angles are equal to one another.

8. Two Right-lines AC, BD in the ſame Plane, not parallel to each other, being infinitely produced towards thoſe Parts where their Diſtance is leaſt, will interſect each other.





9. If two Right-lines CA, CB, making an Angle C, be reſpectively equal to two other Right-lines FD, FE, making an Angle F, and the Angles which they make C and F be likewiſe equal, the Right-lines AB and DE, joining their Extremes will be equal; alſo the Angle A equal to D, the Angle B equal to E, and the Triangle ABC equal to the Triangle DEF.

If this Axiom ſhould not appear ſufficiently evident; conceive the Triangle DFE to be removed, and ſo apply'd to the Triangle ABC that the Point F may coincide with C, and the Side FD fall upon the Side CA; then, becauſe FD is ſuppoſed equal to CA, the Point D will alſo fall upon B: And the Angle F being equal to the Angle C, the Side FE will fall upon CB; and conſequently the Point E upon the Point B, becauſe FE is ſuppoſed equal to CB. Therefore, ſeeing the Point F coincides with C, D with A, and E with B, it is manifeſt that the two Triangles agree in all their Parts, and are therefore equal and alike in all Reſpects.

NOTES and OBSERVATIONS, with the Signification of Signs and Characters uſed in this Tract.

A Propoſition is, when ſomething is, either, propoſed to be done or demonſtrated.

A Problem is, when ſomething is propoſed to be done.

A Theorem is, when ſomething is propoſed to be demonſtrated.

A Lemma is, when ſome Premiſe is demonſtrated, in order to render what follows and what was firſt intended, the more eaſy.

A Corollary is, a conſequent Truth, gained from ſome preceding Truth or Demonſtration.

A Scholium is, when Remarks or Obſervations are made upon ſomething going before.

The Signification of SIGNS.

The Sign =, denotes that the Quantities betwixt which it ſtands are equal.

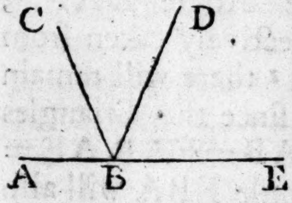
The Sign ⊏, denotes that the Quantity preceding it is greater than that which comes after it.

The Sign ⊐, denotes that the Quantity preceding it is leſs than that which comes after it.

The Sign +, denotes that the Quantity which it precedes is to be added.

The Sign —, denotes that the Quantity which it precedes is to be taken away or ſubſtracted.

When a Figure or Number is prefixt to any Quantity, it ſhews how often that Quantity is to be taken or repeated; as 5A ſhews the Quantity repreſented by A is to be taken 5 Times.



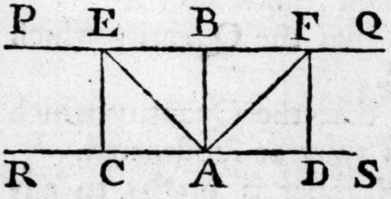
When ſeveral Angles are formed about the ſame Point (as at B) each particular Angle is deſcribed by three Letters, whereof the middle one ſhews the angular Point, and the other two, the Lines that form the Angle: Thus CBD or DBC ſignifies the Angle formed by the Lines CB and DB.

When in any Demonſtration, you meet with ſeveral Quantities joined the one to the other continually by the Mark of Equality (=), the Concluſion drawn from thence is always gathered from the firſt and laſt of them; which are equal to each other, by vertue of the firſt Axiom.

Alſo, when in the Quotations you meet with two Numbers, the firſt ſhews the Propoſition, and the ſecond the Book. Moreover, Ax. denotes Axiom; Poſt. Poſtulatum; Def. Definition; Hyp. Hypotheſis, *&c. &c.*

THEOREM I.

A Right-line AB perpendicular to one of two parallel Right-lines RAS, is alſo perpendicular to the other PBQ.

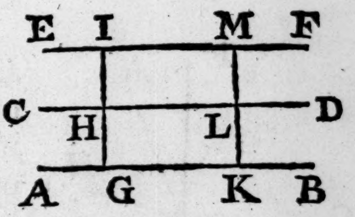


For, in RS let any two equal Diſtances AC, AD be taken [a], alſo let CE and DF be perpendicular to RS [a], and let AE and AF be drawn [b]: Then ſince the Triangles AEC, AFD have AC=AD [c], CE=DF [d] and the Angle ACE=ADF [e], they will alſo have AE=AF, and the Angle CAE = the Angle DAF [f]; which equal Angles being reſpectively taken from the equal Angles CAB, DAB [g] there will remain EAB=FAB [h]: Wherefore, ſince the Triangles EAB, FAB have the Angle EAB=FAB, AE=AF, and AB common, the Angle EBA will alſo be equal to FBA [f] and conſequently AB perpendicular to PQ [k]. *Which was to be demonſtrated.*

[a] Poſt. 4.
[b] Poſt. 1.
[c] Hyp.
[d] Def. 11.
[e] Def. 6, and 11.
[f] Ax. 9.
[g] Ax. 7.
[h] Ax. 5.
[k] Def. 6.

THEOREM II.

Two Right-lines AB, EF parallel to one and the ſame Right-line CD, are parallel to each other.



For, let the Right-lines GHI, KLM be perpendicular to CD meeting AB and EF in the Points G, K, I, M [a]: Then GH being = KL and HI = LM [b], thence will GI = KM [c]; therefore, as both theſe equal Lines are perpendicular to CD [d], they will alſo be perpendicular to AB and EF *(by the precedent)* and conſequently AB and EF parallel to each other [e]. *Which was to be demonſtrated.*

[a] Poſt. 4.

[b] Def. 11.

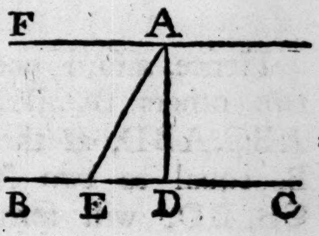
[c] Ax. 4.

[d] Hyp.

[e] Def. 11.

THEOREM III.

From the ſame Point A, to the ſame infinite Right-line BC, can be drawn only one perpendicular Right-line AD.



For, if poſſible, let AE be alſo perpendicular to BC: Then, if AF be conceived parallel to BC [a], the Angle EAF will = a Right-angle [b] = DAF; that is, a Part, equal to the Whole. *Which is impoſſible* [c].

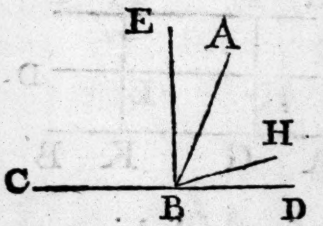
[a] Poſt. 4.

[b] 1. 1. and Ax. 7.

[c] Ax. 2.

THEOREM IV.

A Right-line AB ſtanding upon another Right-line CD makes with it two Angles ABC, ABD, which taken together are equal to two Right-angles.



If the Angles ABC, ABD, are equal, it is plain they make two Right-angles [a], if unequal, let BE be perpendicular to CD [b]; then becauſe $ABC =$ Right-angle $+ ABE$ [c] and $ABD =$ Right-angle $- ABE$ [c], therefore will $ABC + ABD =$ (2 Right-angles $+ ABE - ABE$) 2 Right-angles [d]. *Which was to be demonſtrated.*

[a] Def. 6.
[b] Poſt. 4.
[c] Ax. 3.
[d] Ax. 4.

COROLLARY 1.

Hence, all the Angles at B (made on the ſame Side of the Right-line CD) are equal to two Right-angles [c].

COROLLARY 2.

Hence, alſo, if one Right-line AB, falling upon two others BC, BD makes Angles with them ABC, ABD, at the common Point of Concourſe B, equal to two Right-angles, thoſe two Lines CB, BD, will form one continued Right-line: For, if the contrary was poſſible, then, CB being produced to H, the Angle $ABC + ABH$ would be $= ABC + ABD$ [d], and ſo $ABH = ABD$ [e]: *Which is abſurd* [f].

[d] Ax. 1 and 4. 1.
[e] Ax. 5.
[f] Ax. 2.

THEOREM

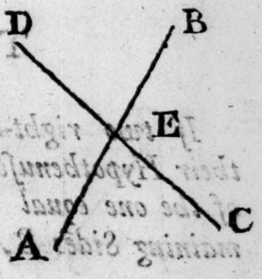
THEOREM V.

The oppoſite Angles AEC, DEB made by two Right-Lines AB, CD interſecting each other, are equal.

For $AED + AEC = 2$ Right-Angles [a] $= AED + DEB$, from which take away AED, common, and there remains $AEC = DEB$ [b]. *Which was to be demonſtrated.*

[a] 4. 1.

[b] Ax. 5.



THEOREM VI.

The Angles A and B at the Baſe of any iſoſceles Triangle ABC are equal to each other.

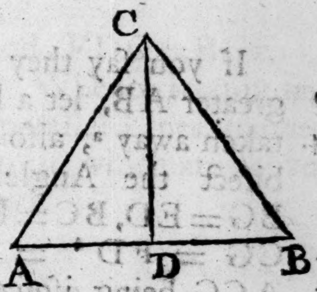
For, let CD biſect, or divide the Angle ACB into two equal Parts [d] and meet AB in D; then the Triangles ACD, BCD having $AC = BC$ [a], CD common, and $ACD = BCD$ [b], will alſo have $A = B$ [c]. *Which was to be demonſtrated.*

[d] Post. 4.

[a] Def. 16.

[b] Conſtr.

[c] Ax. 9.



COROLLARY I.

Hence, it appears that the Right-line which biſects the vertical Angle of an iſoſceles Triangle, biſects the Baſe, and is alſo perpendicular to it [c].

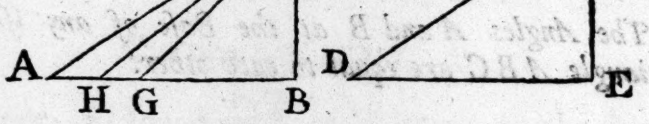
COROLLARY 2.

It appears, moreover, that every equilateral Triangle is likewiſe equiangular [e].

[e] Ax. 1.

THEOREM VII.

If two right-angled Triangles ABC, DEF have their Hypothenuſes AC, DF equal, and alſo one Side BC of the one equal to one Side EF of the other, the remaining Sides BA, ED will likewiſe be equal.



If you ſay they are not equal: Then, from the greater AB, let a Part BG, equal to the leſſer, be taken away [a], alſo let CG be drawn, and let CH biſect the Angle GCA [a]. Therefore, becauſe BG = ED, BC = EF [b] and B = E [c], thence will CG = FD [d] = AC [e], and ſo, the Triangle ACG being iſoſceles [f], HC (as well as BC) will be perpendicular to AB [g]: *Which is impoſſible* [h].

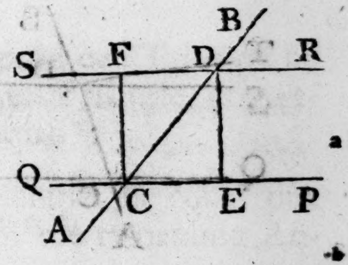
[a] Poſt. 4.
[b] Hyp.
[c] Ax. 7.
[d] Ax. 9.
[e] Hyp. and Ax. 1.
[f] Def. 16.
[g] Cor. 1. to 6. 1.
[h] 3. 1.

COROLLARY.

Hence, becauſe DE = AB, therefore is the Angle D = A, the Angle F = ACB, and the Triangle DEF = the Triangle ABC [d].

THEOREM VIII.

A Right-line AB interſecting two parallel Right-lines PQ, RS, makes the alternate Angles PCD, SDC equal to each other.



Let CF and DE be perpendicular to PQ, and they will alſo be perpendicular to RS [a]; therefore, ſince the Triangles CED, DFC are right-angled at E and F, and have ED = FC [b] and CD common, they will alſo have the Angle ECD = the Angle FDC [c]. *Which was to be demonſtrated.*

[a] 1. 1.
[b] Def. 11.
[c] Cor. to 7. 1.

COROLLARY. 1.

Becauſe BDR = FDC [d], and FDC = BCE [e], therefore is BDR = BCE [f]: Whence it appears that a Line falling upon two parallel Lines, makes the Angles on the ſame Side of it, which are above thoſe Lines, equal to each other.

[d] 5. 1.
[e] 8. 1.
[f] Ax. 1.

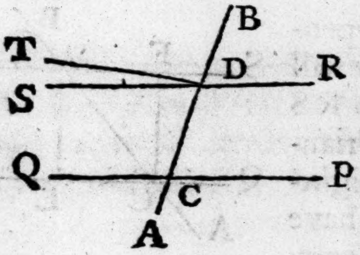
COROLLARY 2.

Moreover, becauſe ECD = FDC [c], and FDC + RDC = 2 Right-angles [g], therefore ECD + RDC is alſo = 2 Right-angles [h]; that is, a Right-line falling upon two parallel Lines, makes the two internal Angles, on the ſame Side, equal to two Right-angles.

[g] 4. 1.
[h] Ax. 4.

THEOREM IX.

If a Right-line AB, falling upon two other Right-lines PQ, RS, makes the alternate Angles PCD, SDC equal to each other, then are thoſe two Lines PQ, RS parallel one to the other.



For, if poſſible, let ſome other Line DT (and not DS) be parallel to PQ, then SDC = DCP [a] = TDC [b]: *Which is impoſſible* [c].

a Hyp.
b 8. 1.
c Ax. 2.

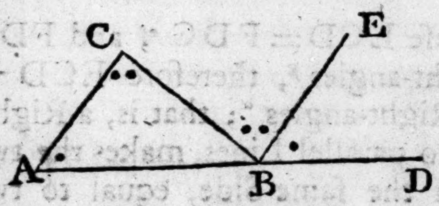
COROLLARY.

Hence, if the Angles BDR, BCP be equal to each other, the Lines RS and PQ will be parallels; becauſe BDR being = SDC [d], BCP will alſo = SDC [e].

d 5. 1.
e Ax. 1.

THEOREM X.

If one Side AB of a Triangle ABC be produced, the external Angle CBD will be equal to both the internal oppoſite Angles A and C taken together.



For, let BE be parallel to AC;[a] then will the Angle C = CBE [b] and the Angle A = DBE [c]; therefore C + A = CBE + DBE [d] = CBD [e]. *Which was to be demonſtrated.*

a Poſt. 4.
b 8. 1.
c Cor. 1 to 8. 1.
d Ax. 4.
e Ax. 3.

COROL-

COROLLARIES.

1. Hence, the three Angles of any plane Triangle, taken together, are equal to two Right-angles: For, ſince $C+A=CBD$, if CBA be added, there will be $C+A+CBA=CBD+CBA$ [f] = 2 Right-angles [g].

2. Therefore, if two Angles in one Triangle be equal to two Angles in another Triangle, the remaining Angles will alſo be equal [h].

3. If one Angle in one Triangle, be equal one Angle in another, the Sums of the remaining Angles will be equal [h].

4. If one Angle of a Triangle be Right, the other two taken together will be equal to a Right-angle.

5. If one Angle be either right or obtuſe, the other two will be both acute.

6. The external Angle is greater than either of the internal oppoſite Angles.

7. The Angle of an equilateral Triangle is equal to $\frac{1}{3}$ of two Right-angles [k].

8. If the vertical Angles of two iſoſceles Triangles be equal, the Angles at the Baſes will alſo be equal [l].

THEOREM XI.

The four inward Angles of any Quadrilateral ABCD are equal to four Right-angles.

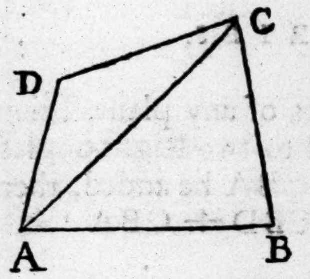
[f] Ax. 4.

[g] Cor. 1. to 4. 1.

[h] Ax. 5.

[k] Cor. 2. to 6. 1.

[l] 6. 1.



Let the Diagonal AC be drawn; then the three Angles of the Triangle ABC being = 2 Right-angles [a], and thoſe of the Triangle ACD equal, alſo, to two Right-angles [a], it follows therefore, that the Sum of all the Angles of both Triangles, which make the four Angles of the Quadrilateral, muſt be equal to four Right-angles [b]. *Which was to be demonſtrated.*

[a] Cor. 1. to 10. 1.

[b] Ax. 4.

COROLLARY 1.

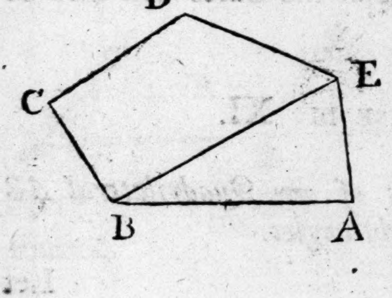
Hence, if three of the Angles be right, the fourth will alſo be right.

COROLLARY 2.

Moreover, if two of the four Angles be equal to two Right-angles, the remaining two will likewiſe be eqaal to two Right-angles.

THEOREM XII.

All the inward Angles of a five-ſided Figure ABCDE, taken together, are equal to ſix Right-angles.



Let BE be drawn. All the Angles of the Quadrilateral BCDE are equal to 4 Right-angles [a], and all the Angles of the Triangle BAE are equal to two Right-angles [b]; therefore all the

[a] 11. 1.

[b] 10. 1.

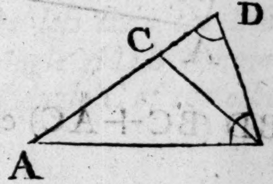
the Angles of the whole Figure ABCDE are equal to ſix Right-angles [c]. *Which was to be demonſtrated.*

SCHOLIUM.

After the ſame Manner it will appear, that the Angles of every ſucceeding Polygon, will exceed the Angles of that immediately preceding it, by two Right-angles.

THEOREM XIII.

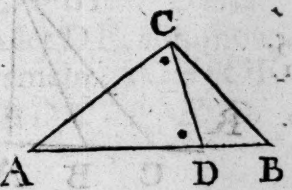
In any Triangle ABC, that Side AB which ſubtends the greateſt Angle ACB is the greateſt.



In AC produced (if need be) let there be taken AD = AB, and let BD be drawn. Then ABC + ACB = ABD + ADB [a] = 2 ABD [b]: But ſince, of the two Angles ABC, ACB, the former ABC is the leſſer [c], it muſt be leſs than half their Sum or ABD [d]; and, ſo BD falling without the Triangle [d], AC [d] is alſo leſs than AD (AB). *Q. E. D.*

THEOREM XIV.

The greateſt Side AB of any Triangle ABC ſubtends the greateſt Angle,



From AB, which is the greateſt, by Hypotheſis, let AD = AC be taken away, and let CD be drawn. The Triangle DAC, being iſoſceles [b], the Angles ACD, ADC will be equal [c]; therefore

---

[c] Ax. 4.

[a] Cor. 3. to 10. 1.

[b] 6. 1.

[c] Hyp.

[d] Ax. 2.

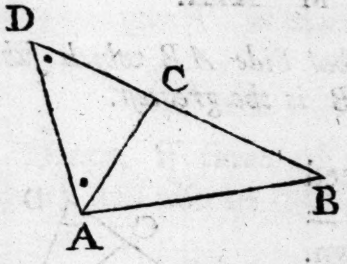
[b] Conſtr.

[c] 6. 1.

fore ACB, which is greater than the former of
d Ax. 2. them [d], muſt alſo be greater than the latter ADC,
e Cor. 6. and conſequently much more greater than B [e], which
to 10. 1. is only a Part of the latter. *Q. E. D.*

THEOREM XV.

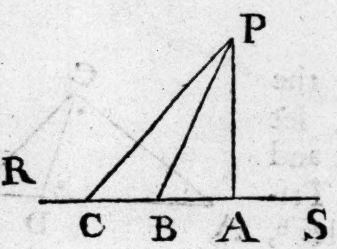
Two Sides AC, BC of a Triangle ABC, any how taken, are greater than the third Side AB.



In BC produced let there be taken CD = CA, and let AD be drawn. The
a 6. 1. Angles D and DAC are equal [a]; therefore BAD,
b Ax. 2. which exceeds the latter [b], muſt alſo exceed the former D, and conſequently
c 13. 1. BD (BC + AC) exceed AB [c]. *Q. E. D.*

THEOREM XVI.

Of all the Right-lines PA, PB, PC falling from a given Point P, upon an infinite Right-line RS, that PA is the leaſt which is perpendicular to RS; and, of the reſt, that PB which is the neareſt the Perpendicular is leſs than any other PC at a greater Diſtance.



a Hyp. For BAP being a Right-angle [a], ABP will be a-
b Cor. 5. cute [b], and therefore AP
to 10. 1. $\sqsupset$ BP [c]. Moreover, ſince
c 13. 1. CBP $\sqsubset$ Right-angle [d], $\sqsubset$ BCP [b], therefore is PC $\sqsubset$ PB [c]. *Q. E. D.*

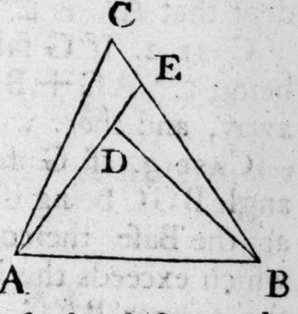
SCHO-

SCHOLIUM.

Becauſe no two Right-lines on the ſame Side of the Perpendicular AP, can be equal to each other (by the Precedent), it is evident that three equal Right-lines cannot be drawn from the Right-line RS to one and the ſame Point P. Whence the Reaſon why the Circumference of a Circle cannot be deſcribed through three Points in the ſame Right-line, is manifeſt.

THEOREM XVII.

If two Triangles ABC, ABD ſtand upon the ſame Baſe AB, and one of them ABD be wholly included within the other ABC, the Sides AD, BD of the included one taken together will be leſs than the Sides AC, BC of the other, but the Angle ADB contained by them, greater than the vertical Angle C of the other.



Let AD be produced to meet BC in E. The Angle ADB, being external, is $\sqsubset$ BED[a], and for the ſame Reaſon is BED $\sqsubset$ C; conſequently ADB $\sqsubset$ C[b]. Again, ſince $AC+CE \sqsubset (AE)\ AD+DE$[c], and $BE+DE \sqsubset BD$[c], by adding the greater to the greater, and the leſs to the leſs, we ſhall have $AE+CE+BE+DE \sqsubset AD+DE+BD$[d], from which let DE, common, be taken away, and there will remain $AC+CE+BE\ (AC+BC) \sqsubset AD+BD$[e]. *Q. E. D.*

[a] Cor. 6. to 10. 1.

[b] Ax. 2.

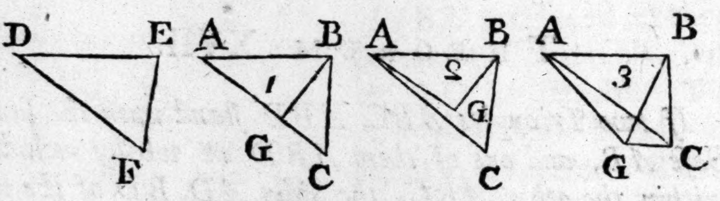
[c] 15. 1.

[d] Ax. 2.

[e] Ax. 6.

THEOREM XVIII.

If two Triangles ABC, DEF have two Sides AB, BC of the one, equal to two Sides DE, EF of the other, each to each reſpectively, and the Angle ABC contained by the two former be greater than the Angle E, contained by the two latter; then, accordingly, the Baſe AC of the former will be greater than the Baſe DF of the latter.



Let the Angle ABG = E, and BG = EF (BC), and let AG and CG be drawn[a].

CASE 1. If the Point G falls in AC; then becauſe ABC $\sqsubset$ ABG (E), by Hypotheſis, it is evident that AC is $\sqsubset$ AG[b].

CASE 2. If G fall above AC; then AB + BC being $\sqsubset$ AG + BG[c], let BC = BG[d] be taken away, and there will remain AC $\sqsubset$ AG[e].

CASE 3. If G falls below AC; then the Triangle BGC being iſoſceles[d], the Angles BCG, BGC at the Baſe thereof are equal[f]; therefore AGC which exceeds the latter of them, muſt alſo exceed the former BCG, and conſequently much more exceed ACG, being only a Part of the former: Whence alſo the Side AC muſt exceed the Side AG[g]. But DF is = AG[h]; therefore AC being *in every Caſe*, greater than AG, it muſt likewiſe be greater than its Equal DF[k]. *Q. E. D.*

[a] Poſt. 1. and 4.

[b] Ax. 2.

[c] 17. 1.

[d] Hyp.

[e] Ax. 6.

[f] 6. 1.

[g] 13. 1.

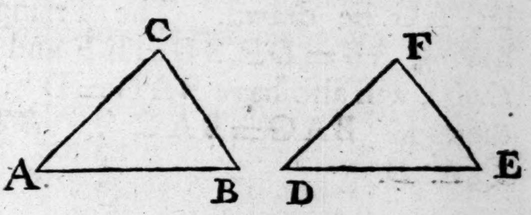
[h] Conſtr.

[k] Ax. 2.

THEO-

THEOREM XIX.

If the three Sides AB, BC, AC of one Triangle, be equal to the three Sides DE, EF, DF of another Triangle, each to each reſpectively, the Angles contained by the equal Sides will alſo be equal.



For, if it were poſſible for any Angle C to be greater or leſs than its correſponding Angle F, then *(by the Precedent)* the oppoſite Side AB would alſo be greater or leſs than the oppoſite Side DE. *Which is contrary to Hypotheſis.*

COROLLARY 1.

Hence, Triangles mutually equilateral, are likewiſe mutually equiangular.

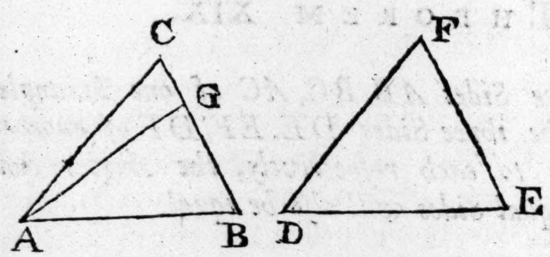
COROLLARY 2.

Hence, alſo, Triangles mutually equilateral, are equal among themſelves [p].

[p] Ax. 9.

THEOREM XX.

If in equiangular Triangles ABC, DEF, two Sides AB, DE, lying between equal Angles, be equal to each other, the other correſponding Sides will alſo be equal.



If you ſay BC is greater than EF; from BC let a Part BG be taken = EF [a], and let AG be drawn. The Triangles ABG, DEF having AB = DE, BG = EF and B = E (*by Hypotheſis*), will alſo have BAG = D [b]; but D = BAC [c], therefore BAG = BAC [d]. *Which is impoſſible.* *Q. E. D.*

[a] Poſt. 4.
[b] Ax. 9.
[c] Hyp.
[d] Ax. 1.

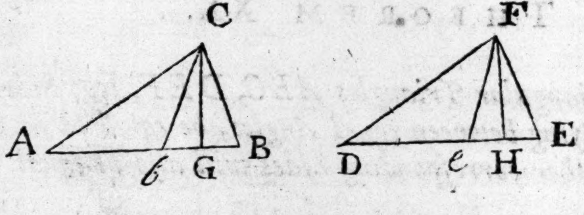
COROLLARY.

Hence, equiangular Triangles, having any two correſponding Sides equal, are equal to each other [e].

[e] Cor. 2. to 19. 1.

THEOREM XXI.

If, in two Triangles ABC, DEF, the two Sides AC, BC of the one be reſpectively equal to the two Sides DF, EF of the other, and the Angles A and D, at the Baſes, ſubtended by the equal Sides BC, EF, likewiſe equal, and the other Angles B and E, at the Baſes, either both acute or both obtuſe; then will the two Triangles be equal in all Reſpects.



Let the Angles B and E be both acute, and let CG and FH be perpendicular to AB and DE. It is evident that the

the Perpendicular CG falls within the Triangle, becauſe the Angle AGC (being ⊏ B, [a]) muſt be external to the Triangle CGB [b]. And the ſame is evident with Regard to the Perpendicular FH. Now, becauſe A = D [a], AGC = DHF [c] and AC = DF [a], therefore is AG = DH and CG = FH [d]; whence, CB being = FE [a], BG is alſo = EH [e], and conſequently AB (AG + BG) = DE (DH + EH [f]:) Therefore, ſeeing the Triangles, ABC, DEF are mutually equilateral, they are equal in all Reſpects [g].

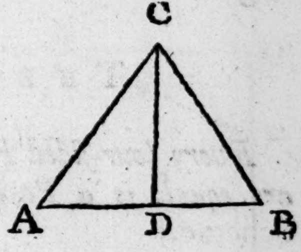
In the ſame Manner the Propoſition may be demonſtrated when both the Angles at the Baſes are obtuſe, as in the Triangles ACb and DFe: For, if Cb = Fe; then, CG being = FH (as above) Gb will be = He [e], and conſequently Ab (AG − Gb) = DE (DH − He, [h]). *Q. E. D.*

[a] Hyp.
[b] Cor. 6 to 10. 1.
[c] Ax. 7.
[d] 20. 1.
[e] 7. 1.
[f] Ax. 4.
[g] Cor. to 19. 1.
[h] Ax. 5.

THEOREM XXII.

If two Angles A and B of a Triangle ABC be equal, the Sides BC, AC ſubtending them will likewiſe be equal.

Let CD biſect the Angle ACB, and meet AB in D: Then the Triangles ACD, BCD being equiangular [a], and having CD common, will alſo have AC = BC [b]. *Q. E. D.*



[a] Hyp. and 10. 1.
[b] 20. 1.

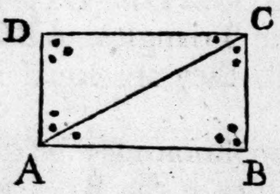
COROLLARY.

Hence, every equiangular Triangle is alſo equilateral.

THEOREM

THEOREM XXIII.

The oppoſite Sides AB, DC and the oppoſite Angles B, D of any Parallelogram ABCD are equal, and the Diagonal AC divides the Whole into two equal Parts.



For, AB, DC, and AD, BC being Parallels [a], the Angle BAC is = DCA [b] and BCA = DAC [b]; therefore the equiangular Triangles ABC, ADC [c], having AC common, are equal in all Reſpects [d]. *Q. E. D.*

[a] Def. 22.
[b] 8. 1.
[c] 10. 1.
[d] 20. 1.

COROLLARY.

Hence, if one Angle B of a Parallelogram be a Right-angle, all the other three will be right: For D, being = B, is manifeſtly a Right-angle; and therefore the Sum of the other two equal Angles BAD, BCD being equal to two Right-angles *(by Cor. 2. to Theo. 11.)* they muſt, both of them be Right.

THEOREM XXIV.

Every four-ſided Figure ABCD whoſe oppoſite Sides are equal, is a Parallelogram. (See the preceding Scheme.)

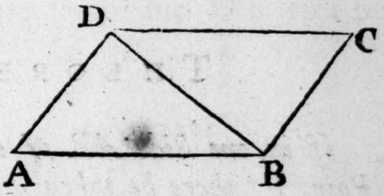
Let the Diagonal AC be drawn, then the Triangles ABC, ADC being mutually equilateral [a], will alſo be mutually equiangular [b], and conſequently AB parallel to DC, and AD to BC [c]. *Q. E. D.*

[a] Hyp.
[b] Cor. 1. to 18. 1.
[c] 9. 1.

THEO-

THEOREM XXV.

The Lines AD, BC joining the Extremes of two equal, and parallel Lines AB, DC (and not croſſing each other) are themſelves equal and parallel.

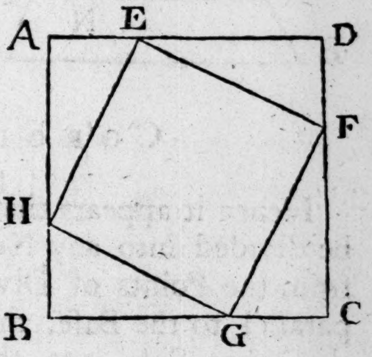


Let the Diagonal BD be drawn. Becauſe AB and DC are parallel [a], the Angle ABD is = CDB [b]; therefore BA being = DC [a], and BD common, the remaining Sides and Angles will likewiſe be equal [c], and conſequently AD parallel to BC [d]. *Q. E. D.*

[a] Hyp.
[b] 8. 1.
[c] Ax. 9.
[d] 9. 1.

THEOREM XXVI.

If in the Sides of a Square ABCD, equally diſtant from the four angular Points, there be taken four other Points E, F, G, H, the Figure EFGH formed by joining thoſe Points ſhall alſo be a Square.



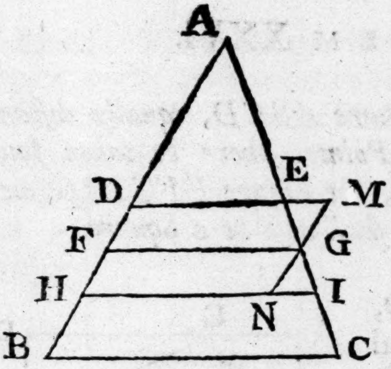
For the Wholes AD, DC, CB, BA being equal [a], and alſo the Parts AE, DF, CG, BH [b], the remaining Parts ED, FC, GB, HA muſt conſequently be equal [c]; whence, all the Angles D, C, B, A being equal [d], the Sides EF, FG, GH, HE will be equal likewiſe [e], and

[a] Def. 24.
[b] Hyp.
[c] Ax. 5.
[d] Ax. 7.
[e] Ax. 9.

e Ax. 9. the Angle DEF = AHE[e]. Therefore, becauſe
f 10. 1. DEH (HEF + DEF) = A + AHE[f], if the
equal Angles DEF, AHE be taken away, there
will remain HEF = A[c] = a Right-angle[d]. Where-
g 24. 1. fore, ſeeing EFGH is a Parallelogram[g] having all
its Sides equal, and one Angle right, the other
h Cor. to 23. 1. three Angles will alſo be Right-Angles[h]. *Q. E. D.*

THEOREM XXVII.

If in one Side AB of a Triangle ABC, from any Point F, there be taken any two equal Diſtances FD and FH, and Right-lines DEM, FG, HI be drawn parallel to the Baſe BC, the Parts GE, GI cut off from the other Side AC, will alſo be equal to each other.



Let NGM be parallel
to AB interſecting HI
and DE in N and M.
Then, the Triangles
IGN, MGE having the
a 5. 1. Angle IGN = EGM[a],
b 8. 1. ING = M[b], and GN
c 23. 1. (= FH,[c] = FD,[d]) =
d Hyp. GM[c] will alſo have
e 20. 1. GI = GE[e]. *Q. E. D.*

COROLLARY I.

Hence it appears that, if one Side of a Triangle be divided into any Number of equal Parts, and from the Points of Diviſion Right-lines be drawn parallel to the Baſe, thoſe Right-lines will divide the other Side into the ſame Number of equal Parts.

COROL-

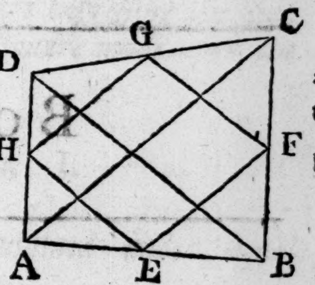
COROLLARY 2.

Hence, alſo, if two Right-lines FG, HI, cutting the Sides of a Triangle, be parallel to each other, and another Line DE be ſo drawn as to cut off FD = FH and GE = GI, that Line DE will be parallel to the two former.

THEOREM XXVIII.

If all the Sides of any Quadrilateral ABCD be biſected, the Figure EFGH formed by joining the Points of Biſection, will be a Parallelogram.

Draw the Diagonals AC and BD. Becauſe EF and HG are parallel to AC [a] they are alſo parallel to each other [b]. After the ſame Manner is FG parallel to EH; therefore EFGH is a Parallelogram [c]. *Which was to be demonſtrated.*



[a] Cor. 1. to 26. 1.

[b] 2. 1.

[c] Def. 22.

The End of the FIRST BOOK.

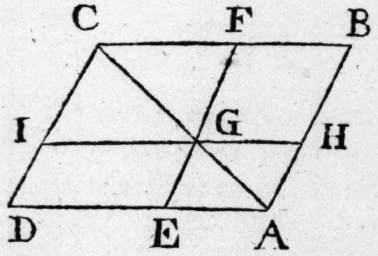
ELEMENTS

OF

Plane Geometry.

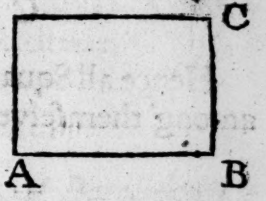
BOOK II.

DEFINITIONS.



1. IN a Parallelogram ABCD, if two Right-lines EF, HI, parallel to the Sides, interſecting the Diagonal in the ſame Point G, be drawn, and thereby the Parallelogram divided into four Parallelograms; theſe two GD, GB through which the Diagonal does not paſs, are called Complements; and the other two HE, FI, Parallelograms about the Diagonal.

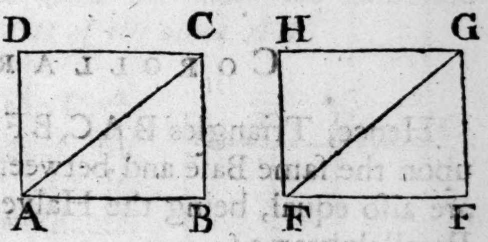
2. Any Rectangle AC is ſaid to be contained under the two Right-lines AB, BC that are the Baſe and Altitude thereof.



The Rectangle contained under two Lines AB and BC (or which they would contain, were they placed at Right-angles, &c.) is often, for Brevity Sake denoted by $AB \times BC$. *Alſo* $\overline{AB + BC} \times \overline{AB - BC}$ *denotes the Rectangle contained under the Lines* $AB + BC$ *and* $AB - BC$; *that is, the Rectangle under the Sum and Difference of the Lines AB and BC: Underſtand the like of others, when they occur. But when the Figure is a Square, it is uſually repreſented by placing the Figure 2 over the Letter or Letters expreſſing the Side thereof: Thus* AB^2 *denotes the Square made upon the Line AB.*

THEOREM I.

The Rectangles BD, FH contained under equal Right-lines, are equal.



For, let the Diagonals AC, EG be drawn: Then, becauſe AB=EF, BC=FG and B =F [a], the Triangles ABC, EFG are equal [b]. And, in the very ſame Manner will ADC and EHG appear to be equal: Therefore the whole Rectangle ABCD is alſo equal to the whole Rectangle EFGH [c]. *Which was to be demonſtrated.*

[a] Hyp.

[b] Ax. 9. 1.

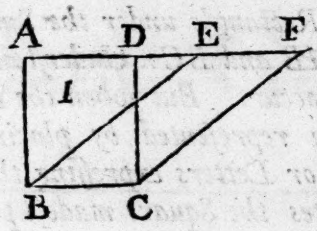
[c] Ax. 4. 1.

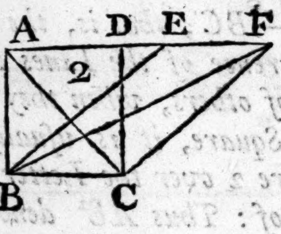
COROLLARY.

Hence all Squares whoſe Sides are equal, are equal among themſelves.

THEOREM II.

Parallelograms ABCD, BCFE, ſtanding upon the ſame Baſe BC and between the ſame Parallels BC, AF, are equal.





a Cor. 1. For, ſince (in *Fig.* 1.) the Angle F=BEA[a],
to 8. 1. and CDF=A[a], the Triangles FDC, EAB are
c 10. 1. equiangular[c]; they are alſo equal[e], becauſe CF=
d 23. 1. BE[d]: Therefore, if each be taken from the whole
e Cor. to Figure ABCF, there will remain ABCD=
20. 1. f Ax. 5. 1. BCFE[f]. *Which was to be demonſtrated.*

COROLLARY 1.

Hence, Triangles BAC, BFC (*Fig.* 2.) ſtanding upon the ſame Baſe and between the ſame Parallels, are alſo equal, being the Halves of their reſpective Parallelograms[f]. f 23. 1.

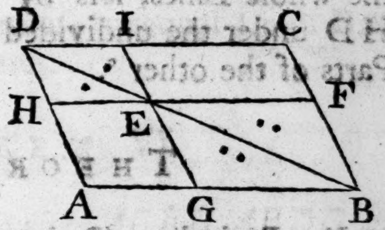
COROLLARY 2.

Hence, and from Theorem 1, it follows, that all Parallelograms, or Triangles whatever, whoſe Baſes and Altitudes are equal, are equal among themſelves; and that every Triangle is equal to half a Rectangle of the ſame Baſe and Altitude.

THEOREM III.

The Complements EC, EA of any Parallelogram AC are equal.

For, the whole Triangle DBC being equal to the whole Triangle DAB [a], and the Parts DIE, EFB reſpectively equal to the Parts DEH, EGB [a], the remaining Parts EC, EA muſt likewiſe be equal [b]. *Which was to be demonſtrated.*



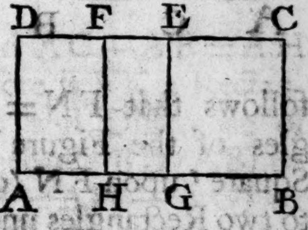
[a] 23. 1. and Def. 1.

[b] Ax. 5. 1.

THEOREM IV.

If two Right-lines AB, BC be given, and one of them AB be divided into any Number of Parts, AH, HG, GB, either equal or unequal, the Rectangle contained under the two whole Lines AB, BC, will be equal to all the Rectangles contained under the undivided Line BC, and the Parts of the other AB.

Let ABCD be the Rectangle contained under the two whole Lines, and let HF, GE be perpendicular to AB, meeting DC in F and E. Then will BE, GF, HD be Rectangles [a] of the ſame Altitude with BD [b]; therefore AF=BC×AH, HE=BC×HG and GC=BC×BG [c] and con-



[a] Cor. to 23. 1.

[b] 23. 1. and Ax. 1.

[c] 1. 2.

conſequently $BC \times AB\ (= AC = AF + HE + GC) = BC \times AH + BC \times HG + BC \times BG$[d]. *Which was to be demonſtrated.*

[d] Ax. 3. and 4. 1.

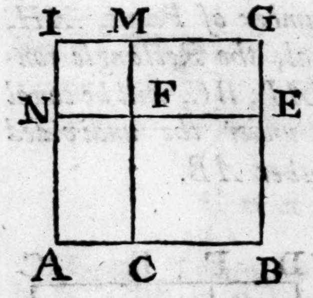
COROLLARY.

Hence, the Rectangle BE, under the whole undivided Line BC and one Part BG of the other, is equal to the Rectangle BD contained under both the whole Lines, leſs by all the Rectangles GF, HD under the undivided Line and the remaining Parts of the other[e].

[e] Ax. 3. and 5. 1.

THEOREM V.

If a Right-line AB be, any wiſe, divided into two Parts AC, BC, the Square of the whole Line, will be equal to the Squares of both the Parts, together with a double Rectangle under the ſame Parts.



Let ABGI be the Square of AB, and CBEF that of BC, and let EF and CF be produced to meet the Sides of the Square ABGI in M and N.

Becauſe $AB = BG$ and $BC = BE$[a], therefore $AC = EG$[b]; whence FN being $= AC$, and $FM = EG$[c] it follows that $FN = FM$[d]; therefore, all the Angles of the Figure being right ones[e], NM is a Square[f] upon FN (or AC) and AF, FG are equal to two Rectangles under BC and AC[g]: But $AG = BF + FI + AF + FG$, or $AB^2 = BC^2 + AC^2 + 2\,AB \times BC$[h]. *Q. E. D.*

[a] Def. 24. 1.
[b] Ax. 5. 1.
[c] 23. 1.
[d] Ax. 1. 1.
[e] Cor. to 23. 1.
[f] Def. 24. 1.
[g] 1. 2.
[h] Ax. 3. 1.

COROL-

COROLLARY I.

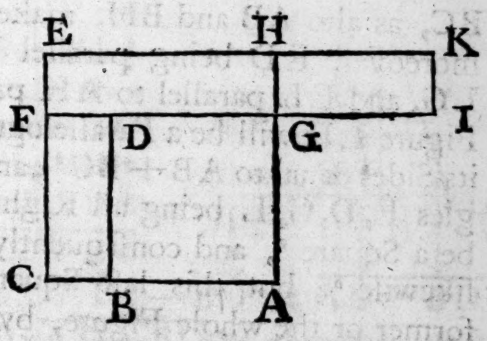
Hence, the Square of any Line is equal to four Times the Square of half that Line.

COROLLARY 2.

Hence, alſo, if two Squares be equal, their Sides muſt be equal; becauſe unequal Lines BA, BC have not equal Squares.

THEOREM VI.

The Difference of the Squares ACEH, ABDG of any two unequal Right-lines AC, AB, is equal to a Rectangle under the Sum and Difference of the ſame Lines.



In EH produced take $HK=AB$; make KI parallel to EC, and let DG be produced both Ways to meet IK and EC in I and F.

Becauſe $AC+AB=EH+HK$ [a] $=EK$, and $AC-AB=AH-AG$ [c] $=HG=KI$ [d], therefore is FK a Rectangle contained under the Sum and Difference of AC and AB [e]: Moreover, becauſe BD (AB) $=HK$, and BC $(=AC-AB=AH-AG)=HG$, the Rectangles BF and GK, being contained under equal Right-lines, will be equal [f]; to which let FH, common, be added, and then $BF+FH$ (=

[a] Ax. 4. 1.

[c] Ax. 5. 1

[d] 23. 1.

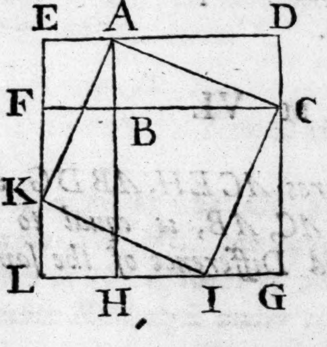
[e] Def. 2. 2.

[f] 1. 2.

(= the Difference of the two Squares AE and AD) = FK[g]. *Which was to be demonſtrated.*

THEOREM VII.

The Square upon the Right-line AC, ſubtending the Right-angle of a plane Triangle ABC, is equal to both the Squares BE, BG made upon the Sides AB, BC including that Angle.



Let the Sides of the Squares BE, BG be produced to meet each other in L and D; in which take KL and IG each equal to AE (or AB) and let CI, IK, and KA be drawn.

Becauſe all the Angles at B are Right-ones[a], FB and BC, as alſo AB and BH, make one Right-line[b]; moreover, ED being parallel to FC parallel to LG, and EL parallel to AH parallel to DG[c], the Figure LD will be a Parallelogram, having each of its Sides equal to AB + BC[d]; and therefore, its Angles E, D, G, L being all Right-ones[e], it will alſo be a Square[g], and conſequently ACIK a Square likewiſe[h]; but this laſt Square is leſs than the former or the whole Figure, by the four Triangles ADC, CGI, ILK, KEA, which are equal among themſelves[i], and whereof each is = half the Rectangle BD or BL[k]: Therefore, ſeeing the Square ACIK is leſs than the whole Figure by a Quantity equal to the two Rectangles BD, BL, and that the two Squares BE, BG are alſo leſs than the whole Figure by the ſame two Rectangles, it is plain that the Square ACIK is equal to both the Squares BE and BG taken together[l]. *Which was to be demonſtrated.*

---

[g] Ax. 4. 1.

[a] Def. 18. and 24. 1.

[b] Cor. 2. to 4. 1.

[c] 2. 1. and Def. 22.

[h] 23. 1.

[e] Cor. to 23. 1.

[g] Def. 24. 1.

[b] 26. 1.

[i] Ax. 9. 1.

[k] Cor. 2. to 2. 2.

[l] Ax. 5. 1.

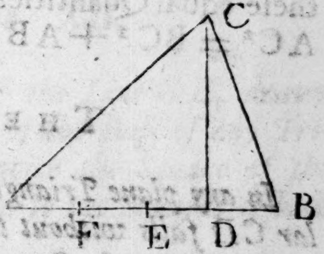
COROLLARY.

Hence, the Square upon either of the Sides, including the Right-angle, is equal to the Difference of the Squares of the Hypothenuſe and the other Side [a]; or equal to a Rectangle contained under the Sum and Difference of the Hypothenuſe and the other Side [b].

[a] Ax. 5. 1.
[b] 6. 2.

THEOREM VIII.

In any Triangle ABC, where the Perpendicular CD, from the Vertex to the Baſe, falls within the Triangle, the Rectangle, contained under the Sum and Difference of the two Sides AC and BC, is equal to the Rectangle contained under the whole Baſe AB, and the Difference of its two Segments AD and BD.



For, $AC^2 = AD^2 + DC^2$ [a], and $BC^2 = BD^2 + DC^2$ [a]; therefore, by taking the latter from the former, we have $AC^2 - BC^2 = AD^2 - BD^2$ [b]; but $AC^2 - BC^2 = \overline{AC+BC} \times \overline{AC-BC}$ [c], and $AD^2 - BD^2 = \overline{AD+BD} \times \overline{AD-BD} = AB \times \overline{AD-BD}$ [c]; therefore, alſo, $\overline{AC+BC} \times \overline{AC-BC} = AB \times \overline{AD-BD}$ [d]. *Q. E. D.*

[a] 7. 2.
[b] Ax. 5. 1.
[c] 6. 2.
[d] Ax. 1. 1.

COROLLARY.

Hence, if AF be taken = BD, and AB be biſected in E, then AD − BD being = AD − AF [a],

[a] Ax. 5. 1.

[b] Hyp.
[c] Ax. 1. 1.

$= FD = 2ED$ [b], it follows that $\overline{AC+BC}\times AC-BC = AB\times 2ED$ [c]; or that, the Rectangle under the Sum and the Difference of the Sides is equal to the Rectangle under the whole Baſe and twice the Diſtance of the Perpendicular from the Middle of the Baſe.

THEOREM IX.

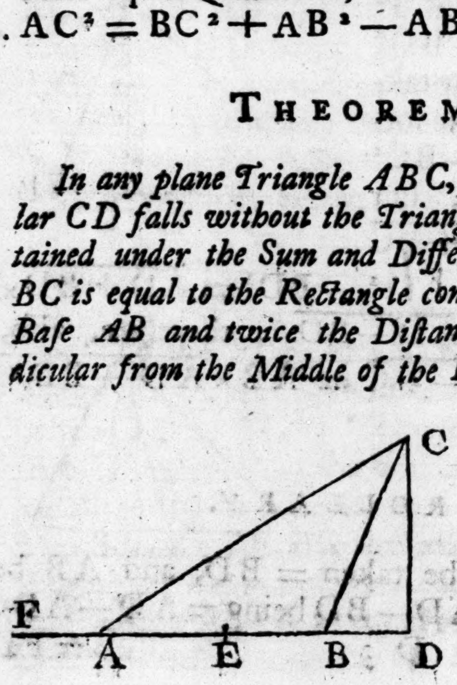
In any plane Triangle ABC, where the Perpendicular falls within the Triangle, the Square of one Side AC is leſs than the Sum of the Squares of the Baſe and other Side, by a double Rectangle under the whole Baſe AB and the alternate Segment BD. (See the preceding Scheme.)

[a] 6. 2.
[b] 8. 2.
[c] Cor. to 4. 2.
[d] Ax. 4. 1.

For, becauſe $AC^2 - BC^2 = \overline{AC+BC}\times\overline{AC-BC}$ [a] $= AB\times DF$ [b] $= AB\times\overline{AB-2BD}$ [c] $= AB^2 - AB\times 2BD$ [c], to the firſt and laſt of theſe equal Quantities, let BC^2 be added, and then $AC^2 = BC^2 + AB^2 - AB\times 2BD$ [d]. *Q. E. D.*

THEOREM X.

In any plane Triangle ABC, where the Perpendicular CD falls without the Triangle, the Rectangle contained under the Sum and Difference of the Sides AC, BC is equal to the Rectangle contained under the whole Baſe AB and twice the Diſtance ED of the Perpendicular from the Middle of the Baſe.



[a] 7. 2.

In BA produced let AF be taken equal to BD. Becauſe $AC^2 = AD^2 + DC^2$ [a] and $BC^2 = BD^2 + DC^2$ [a], thence will $AC^2 - BC^2$

$-BC^2 = AD^2 - BD^2$ [b]; but $AC^2 - BC^2 = \overline{AC+BC} \times \overline{AC-BC}$ [c] and $AD^2 - BD^2 = \overline{AD+BD} \times \overline{AD-BD}$ [c] $= FD \times AB$ [d] $= 2\,ED \times AB$ [e]; therefore $\overline{AC+BC} \times \overline{AC-BC} = 2\,ED \times AB$ [f]. *Q. E. D.*

[b] Ax. 5. 1.
[c] 6. 2.
[d] 4. 2.
[e] Ax. 3. 1.
[f] Ax. 1. 1.

THEOREM XI.

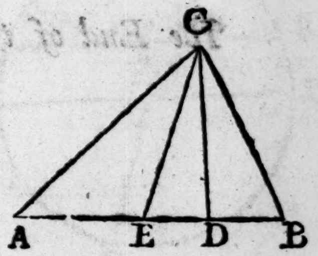
The Square of the Side AC, ſubtending the obtuſe Angle B of a Triangle ABC, is equal to the Sum of the Squares of the other Sides AB and BC, together with a double Rectangle under the whole Baſe AB and the Diſtance BD of the Perpendicular from the obtuſe Angle. (See the preceding Figure.)

For, $AC^2 = AD^2 + DC^2$ [a] $= AB^2 + BD^2 + 2\,AB \times BD + DC^2$ [b] $= AB^2 + BC^2 + 2\,AB \times BD$ [c]. *Q. E. D.*

[a] 7. 2.
[b] 5. 2. and Ax. 4. 1.
[c] 7. 2. and Ax. 4. 1.

THEOREM XII.

The double of the Square of the Line CE, drawn from the Vertex to the Middle of the Baſe of any Triangle ABC, together with twice the Square of the Semi-baſe AE (or BE), is equal to the Squares of both the two Sides AC, BC.

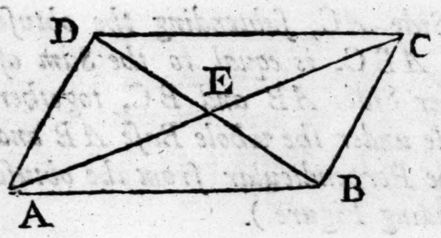


For let CD be perpendicular to AB: Then $AC^2 = AE^2 + EC^2 + 2\,AE \times ED$ [a], and $BC^2 = BE^2\ (AE^2) + EC^2 - 2\,BE\ (2\,AE) \times ED$ [b]; whence, by adding together theſe equal Quantities, we have $AC^2 + BC^2 = 2\,AE^2 + 2\,EC^2$ [c]. *Q. E. D.*

[a] 11. 2.
[b] 9. 2.
[c] Ax. 4. 1.

THEOREM XIII.

The Diagonals AC, BD of a Parallelogram ABCD biſect each other; and the Sum of their Squares is equal to the Sum of the Squares of all the four Sides of the Parallelogram.



The Triangles AEB, DEC being equiangular [a], and having AB = DC [b], will alſo have AE = CE and BE = DE [c]. Moreover, becauſe $2\,AE^2 + 2\,ED^2 = AD^2 + DC^2$ [d] and $2\,AE^2 + 2\,ED^2 = BC^2 + AB^2$ [d], therefore $AD^2 + DC^2 + BC^2 + AB^2 = 4\,AE^2 + 4\,ED^2$ [e] $= AC^2 + BD^2$ [f]. *Which was to be demonſtrated.*

[a] 8. 1.
[b] 23. 1.
[c] 20. 1.
[d] 12. 2.
[e] Ax. 4. 1.
[f] Cor. to 5. 2.

The End of the SECOND BOOK.

ELEMENTS

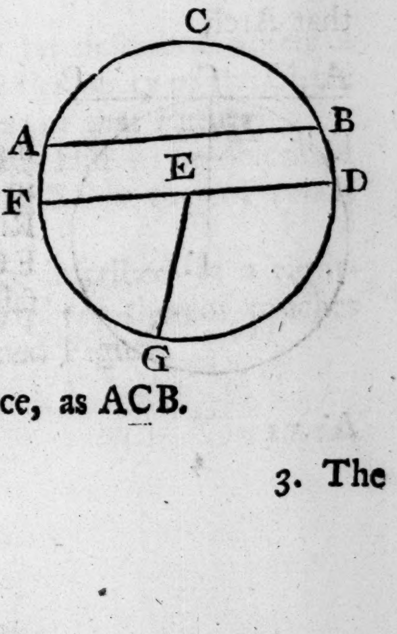
OF

Plane Geometry.

BOOK III.

DEFINITIONS.

1. ANY Right-line FD, paſsing thro' E the Center of a Circle and terminating in the Circumference at both Ends, is called a Diameter.



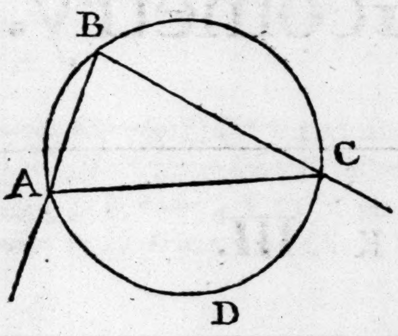
2. An Arch of a Circle is any Portion of the Periphery or Circumference, as ACB.

3. The Chord or Subtenſe of an Arch ACB is a Right-line AB joining the two Extremes of that Arch.

4. A Semicircle is a Figure contained under any Diameter and the Part of the Circumference cut off by that Diameter.

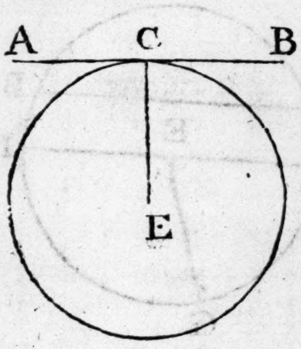
5. A Segment of a Circle is a Figure contained under an Arch ACB and its Chord AB.

6. A Sector of a Circle is a Figure contained under two Right-lines EF, EG, drawn from the Center to the Circumference, and the Arch FG included betwixt them.

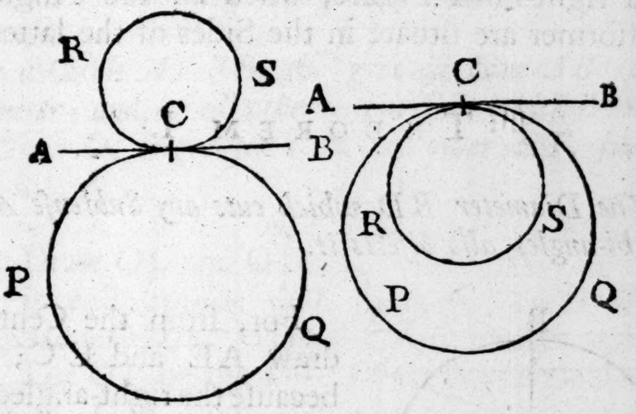


7. An Angle ABC is ſaid to be in a Segment of a Circle ABC, when, being in the Periphery thereof, the Right-lines BA, BC by which it is formed, paſs through the Extremes of the Baſe, or Subtenſe AC of that Segment.

8. An Angle ABC in the Periphery, comprehended by two Right-lines BA, BC, including an Arch of the Circle ADC; is ſaid to ſtand upon that Arch.



9. A Right-line AB meeting the Periphery of a Circle in any Point C, and making Right-angles with the Radius EC drawn to that Point, is ſaid to touch the Circle.



10. Two Circles PCQ, RCS are ſaid to touch each other, when the Right-line AB touching one of them at the Point of Concourſe C, alſo touches the other in the ſame Point.

11. Two Circles are ſaid to cut one another, when they fall partly within, and partly without each other.

12. A Right-line is ſaid to be applied, or inſcribed in a Circle when both its Extremes are in the Periphery thereof.

13. A right-lined Figure is ſaid to be inſcribed in a Circle when all its Angles are in the Circumference of the Circle.

14. A Circle is ſaid to be deſcribed about a right-lined Figure when the Periphery of the Circle paſſes through all the Angles of that Figure.

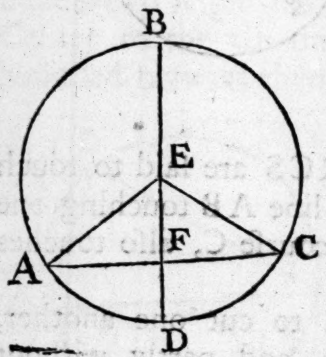
15. A right-lined Figure is ſaid to be deſcribed about a Circle, when all the Sides thereof touch the Circle.

16. A Circle is ſaid to be inſcribed in a right-lined Figure, when the Periphery thereof touches all the Sides of the right-lined Figure.

17. A right-lined Figure is ſaid to be inſcribed in a right-lined Figure, when all the Angles of the former are ſituate in the Sides of the latter.

THEOREM I.

The Diameter BD which cuts any Subtenſe AC at Right-angles alſo biſects it.

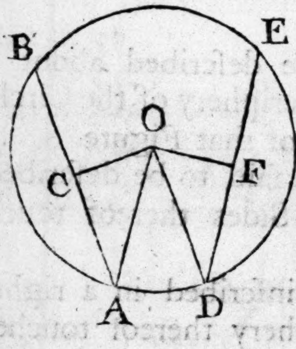


For, from the Center E draw AE and EC; then becauſe the right-angled Triangles AEF, CEF [a] have the ſame Perpendicular FE, and equal Hypothenuſes AE and EC [b], their Baſes AF and CF will alſo be equal [c]. *Which was to be demonſtrated.*

[a] Hyp. and Def. 18. 1.
[b] Def. 29. of 1.
[c] 7. 1.

THEOREM II.

In any Circle ABED, Right-lines AB, DE, equally diſtant from the Center O, are equal to each other.

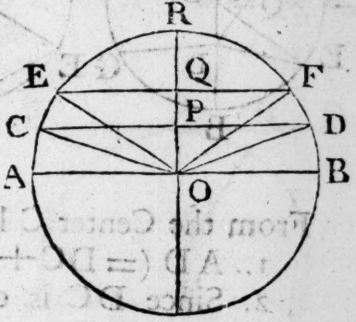


Let the Perpendiculars OF, OC be drawn, and let O, D and O, A be joined. Becauſe OF = OC [a], OD = OA [b], and F and C are both Right-angles [c], therefore is DF = AC [d], and conſequently DE = 2 DF [e] = 2 AC [f] = AB [e]. *Q. E. D.*

[a] Hyp.
[b] Def. 29. of 1.
[c] Conſtr.
[d] 7. 1.
[e] 1. 3.
[f] Ax. 4. 1.

THEOREM III.

In a Circle AEFB, the greateſt Line AB is the Diameter; and, of all others, that CD which is neareſt the Center O, is greater than any other EF, further from it.



1. Draw OC and OD; then it will appear that $AB(OC+OD)\sqsubset CD$[a]. [a] 15. 1.

2. Let OP be the Diſtance of CD from the Center, and OQ that of EF, both taken in the ſame Radius OR; draw OE and OF. Becauſe the Triangles DOC, OFE, have two Sides equal each to each[b] and have the contained Angle DOC $\sqsubset$ the contained Angle FOE[c], therefore alſo will the Baſe DC be greater than the Baſe FE[d], and conſequently greater than any other Right-line at the ſame Diſtance with EF[e]. *Q. E. D.* [b] Def. 29. 1. [c] Ax. 2. 1. [d] 18. 1. [e] 2. 3.

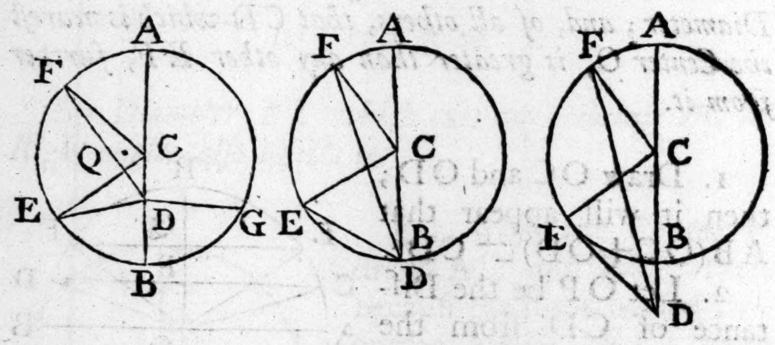
COROLLARY.

Hence, a Right-line greater than the Diameter, drawn from any Point within a Circle, cuts the Circumference; and if any Part of a Right-line, falling within a Circle, be infinitely produced both Ways, it will cut the Circumference in two Points.

THEOREM IV.

If to the Circumference of a Circle AFEBG from any Point D which is not the Center, Right-lines DA, DF, DE be drawn, the greateſt of all DA ſhall be that

that which paſſes through the Center C; and, of the reſt, that DF which is neareſt the greateſt DA ſhall exceed any other DE at a greater Diſtance from it.



From the Center C let CE and CF be drawn.

1. AD ($=$ DC $+$ CF, [b]) $\sqsubset$ DF [c].

2. Since DC is common, CF $=$ CE [d] and DCF $\sqsubset$ DCE [e], therefore is DF $\sqsubset$ DE [f]. *Q. E. D.*

[b] Def. 29. and Ax. 4. 1.
[c] 15. 1.
[d] Def. 29. 1.
[e] Ax. 2. 1.
[f] 18. 1.

COROLLARY 1.

Becauſe no two Lines on the ſame Side the Diameter AB can be equal to each other [a], it is plain that three equal Right-lines cannot poſſibly be drawn from the Periphery to any Point beſides the Center of the Circle: And therefore, if from a Point in any Circle three equal Right-lines can be drawn to the Periphery, that Point is the Center of the Circle.

[a] 4. 3.

COROLLARY 2.

Hence, it alſo follows that no Circle can be deſcribed to cut or touch another FBG in more Points than two: For, if it were poſſible to cut it in three Points G, E, F, then Right-lines drawn from the Center Q to thoſe Points would be all equal,

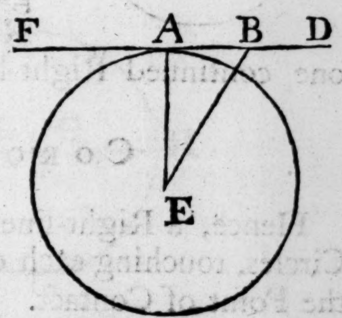
equal [b], which is ſhewn to be impoſſible [c], unleſs when the Centers C and Q coincide, and then it is plain the Circles themſelves would neither cut nor touch but alſo coincide or become one Circle [b].

[b] Def. 29. 1.

[c] Cor. 1. to 4. 3.

Theorem V.

The Tangent FAD, to any Point A of a Circle E, falls wholly without the Circle.



From any Point in the Tangent to the Center E, let BE be drawn and join E, A. Becauſe EA is perpendicular to AB [a], it is therefore leſs than EB [b], and ſo the Point B falls above the Circle [c]. And as the like may be inferred with Regard to any Point in FD (except A) it is manifeſt that the Tangent FD does not cut off any Part of the Circle, but only touch its Periphery in the Point A. *Which was to be demonſtrated.*

[a] Def. 9. 3.

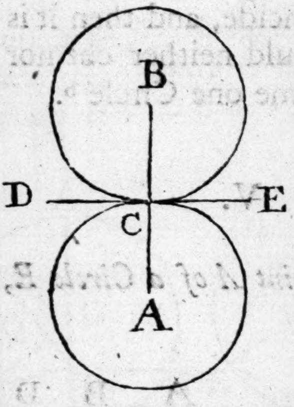
[b] 16. 1.

[c] Def. 29. and Ax. 2. 1.

Theorem VI.

Two Circles A, B touching one another outwardly fall wholly without each other; and the two Radii AC, BC, drawn, from their Centers A, B to the Point of Contact C, make one continued Right-line ACB.

Let



[a] Def. 10. 3.

Let the Right-line DCE be the common Tangent to both Circles at C the Point of Contact [a]. The Circle A falling wholly below, and the Circle B wholly above DCE (*by the precedent*), it is manifeſt that they fall wholly without each other. Moreover, the Angles ACD, BCD being both Right-angles [b], ACB will be one continued Right-line [c]. *Q. E. D.*

[b] Def. 9. 3.

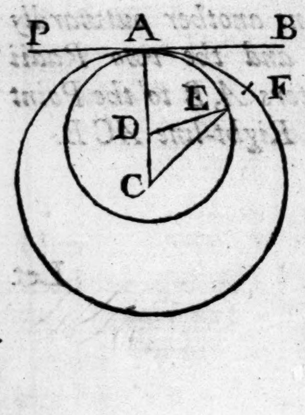
[c] Cor. 2. to 4. 1.

COROLLARY.

Hence, a Right-line joining the Centers of two Circles, touching each other outwardly, paſſes thro' the Point of Contact.

THEOREM VII.

If two unequal Circles AE, AF, touch each other inwardly, the leſſer AE will fall wholly within the greater AF; and a Right-line CA drawn from the Center C of the greater to the Point of Contact A will paſs thro' rhe Center of the leſſer.



[a] Def. 10. 3.

Let PAB be the common Tangent [a]; and from D the Center of the leſſer Circle, to any Point E in the Periphery thereof, let DE be drawn; and join C, E and D, A. The Angles PAD, PAC being both right [b], AD and AC coincide [c], and ſo the Point D falls

[b] Def. 9. 3.

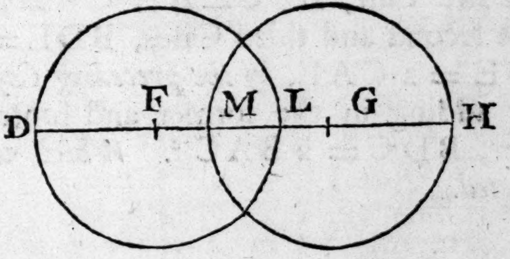
[c] Ax. 7. 1.

falls in the Line AC[d]. Moreover CA is greater than CE[e]; and therefore the Point E falls within the Circle AF[f]. *Q. E. D.*

[d] Ax. 2. 1.
[e] 4. 3.
[f] Def. 29. and Ax. 2. of 1.

THEOREM VIII.

If the Diſtance of the Centers F, G of two Circles DL, MH be leſs than the Sum, and greater than the Difference of their Semidiameters FL, GM, thoſe Circles will cut each other.

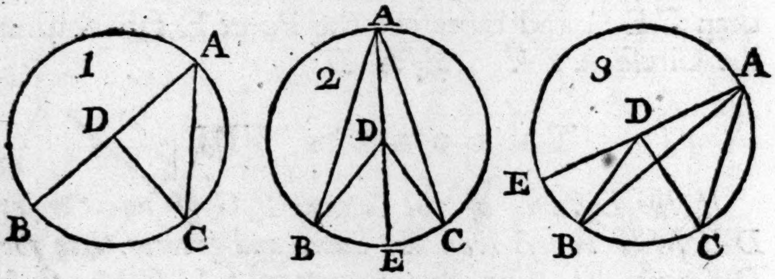


For, let FG, produced both Ways, meet the Peripheries of the two Circles in the Points D, L, M, H[a]. Becauſe $FG \sqsupset FL + GM$[b], if GM, common, be taken away, there will remain $FM \sqsupset FL$[c]; whence it is manifeſt that the Point M falls within the Circle DL[d]. Moreover, becauſe $FG \sqsubset FL - GM$[b], if GM be added, then will $FG + GM$ $(FG + GH) = FH$ be alſo $\sqsubset FL$[c], and ſo the Point H fall without the Circle DL[d]. Therefore, ſeeing the Point M falls within, and the Point H without the Circle DL, the two Circles cut each other[e]. After the ſame Manner the Propoſition may be proved in any other Caſe. *Q. E. D.*

[a] Cor. to 3. 3.
[b] Hyp.
[c] Ax. 6. 1.
[d] Ax. 2. 1.
[e] Def. 11. 3.

THEOREM IX.

The Angle BDC, at the Center of a Circle, is double to the Angle BAC at the Periphery, when both Angles ſtand upon the ſame Arch BC.



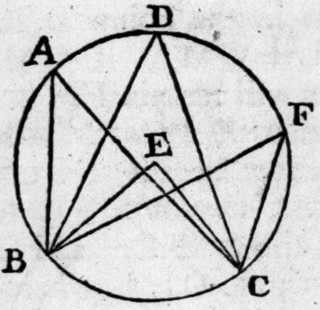
Let the Diameter ADE be drawn.

a 10. 1. In the firſt Caſe, BDC = A + C[a] = 2 A[b].

b 6. 1. In the ſecond and third Caſes, BDE = 2 BAE and CDE = 2 CAE, *by the preceding Caſe*; therefore, by adding in the former and ſubtracting in the latter, BDC = 2 BAC[c]. *Which was to be demonſtrated.*

c Ax. 4. and 5.

COROLLARY.

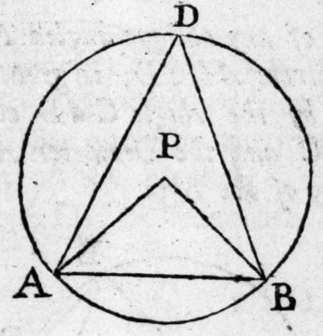


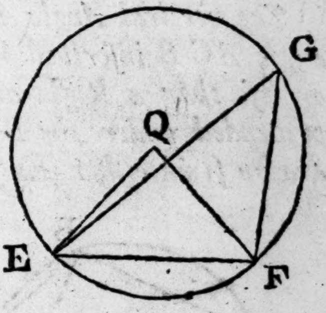
Hence, all the Angles A, D, F, ſtanding upon the ſame Arch of a Circle BC, are equal among themſelves, being each of them equal to half the Angle E at the Center.

THEOREM X.

Angles D, G, in the Periphery, and ſtanding upon equal Subtenſes AB, EF of Circles ADB, EGF whoſe Diameters are equal, are equal to each other. And the Subtenſes of equal Angles, in the Peripheries of Circles whoſe Diameters are equal, are alſo equal.

From





From the Centers P, and Q, let PA, PB, QE, QF be drawn.

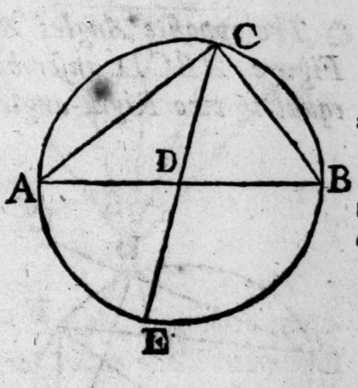
1. Hyp. Since AB = EF [a] and AP = BP [b] = EQ = FQ [a]; therefore is P = Q [c] and conſequently D ($= \frac{1}{2}$ P [d] $= \frac{1}{2}$ Q) = G.

2. Hyp. Becauſe D = G [a], therefore P = Q [d]; whence, PA being = QE, and PB = QF [a], AB will alſo be = EF [e]. *Q. E. D.*

[a] Hyp.
[b] Def. 29. 1.
[c] 19. 1.
[d] 9. 3.
[e] Ax. 9. 1.

THEOREM XI.

The Angle ACB in a Semicircle is a Right-angle.



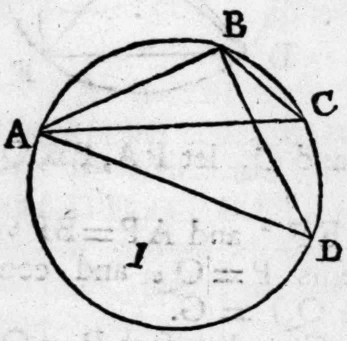
Let the Diameter CDE be drawn.

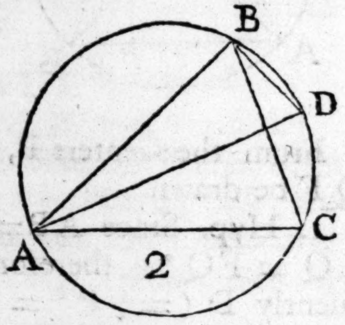
Becauſe ACD = $\frac{1}{2}$ ADE and BCD = $\frac{1}{2}$ BDE [a], therefore is ACD + BCD (ACB) = $\frac{1}{2}$ ADE + $\frac{1}{2}$ BDE [b] = $\frac{1}{2}$ two Right-angles [c] = one Right-angle. *Which was to be demonſtrated.*

[a] 9. 3.
[b] Ax. 4. 1.
[c] 4. 1.

THEOREM XII.

The vertical Angle ABC, of any oblique-angled Triangle ACB inſcribed in a Circle ABCD, is greater, or leſs than a Right-angle, by the Angle CAD comprehended under the Baſe AC and the Diameter AD drawn from either Extremity of the Baſe.



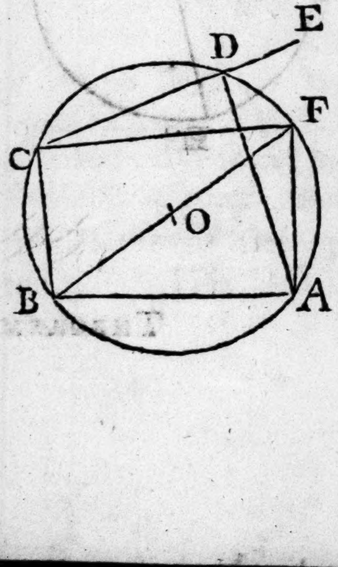


For, BD being drawn, ABD will be a Right-angle [a] and alſo CAD = CBD [b]; therefore, in the firſt Caſe, ABC = Right-angle + CAD, and in the ſecond, ABC = Right-angle — CAD [c]. *Which was to be demonſtrated.*

[a] 11. 3.
[b] Cor. to 9. 3.
[c] Ax. 5. 1.

THEOREM XIII.

The oppoſite Angles ABC, ADC of any four-ſided Figure ABCD inſcribed in a Circle, are together equal to two Right-angles.



Draw the Diameter BF, and join A, F and C, F.

Becauſe the Angles BAF and BCF are both Right-ones [a], therefore ABC and AFC are equal to two Right-angles [b]; but ADC = AFC [c]; therefore ABC + ADC = 2 Right-angles [d]. *Q. E. D.*

[a] 9. 3.
[b] Cor. 2. to 11. 1.
[c] Cor. to 9. 3.
[d] Ax. 4. 1.

COROLLARY 1.

Hence, if one Side of a Quadrilateral, inſcribed in a Circle, be produced, the external Angle EDA will be equal to the internal oppoſite Angle ABC: For ABC + ADC = 2 Right-angles [e] = EDA + ADC [f]; therefore ABC = EDA [g].

[e] 13. 3.
[f] 4. 1.
[g] Ax. 5. 1.

COROLLARY 2.

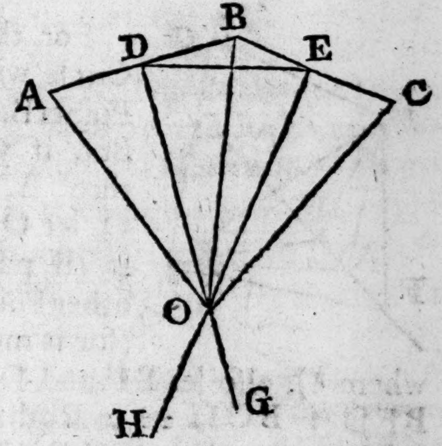
Hence, it alſo appears that no oblique-angled Parallelogram can be circumſcribed by a Circle, becauſe, its oppoſite Angles, being equal to each other [h], are either leſs or greater than two Right-angles.

[h] 23. 1.

THEOREM XIV.

Thro' any three Points A, B, C, not in the ſame Right-line, the Circumference of a Circle may be deſcribed.

Draw AB and BC, which let be biſected by the infinite Perpendiculars DG and EH.



Theſe Perpendiculars, I ſay, will interſect each other, and the Point of Interſection O will be the Center of the Circle.

For, if DE be drawn, it is plain, the Angles GDE, HED are leſs than 2 Right-angles [a]; therefore the Lines DG, EH, being not parallel [b], will meet each other [c]: Wherefore, if

[a] Ax. 2. 1.
[b] Cor. 2. to 8. 1.
[c] x. 8. 1.

if from the Point of Interſection O, the Right-lines OA, OB, OC be now drawn, the Triangles ADO, BDO, having 2 Sides equal each to each [d], and the Angles ADO, BDO contained by them alſo equal [e], will likewiſe have $AO = BO$ [f]: After the very ſame Manner is $CO = BO$; therefore $AO = BO = CO$ [g]: Whence a Circle deſcribed upon the Center O, at the Diſtance of AO, will alſo paſs thro' B and D [h]. *Q. E. D.*

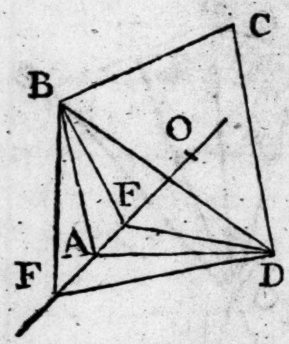
[d] Conſtr.
[e] Ax. 7. 1.
[f] Ax. 9. 1.
[g] Ax. 1. 1.
[h] Def. to 29. 1,

COROLLARY.

Hence, and from Corol. 2 to Theorem 4, it appears that a Perpendicular to the Middle of any Subtenſe paſſes thro' the Center of the Circle.

THEOREM XV.

If the oppoſite Angles BAD, BCD of a Quadrilateral ABCD be equal to two Right-angles, a Circle may be deſcribed about that Quadrilateral.

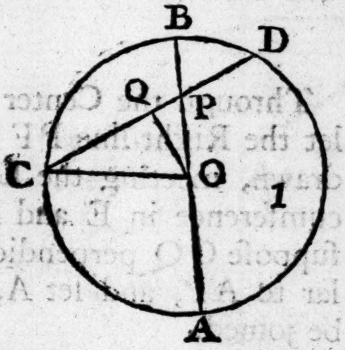


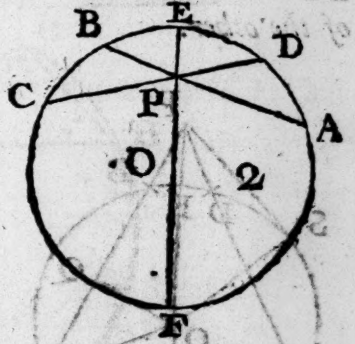
For the Circumference of a Circle will paſs thro' any three Points B, C, D, (*by the precedent.*) But, if you deny that it paſſes thro' A, then thro' the Center O, let OAF be drawn, and let it (if poſſible) paſs thro' ſome other Point F in the Line OAF, (for it muſt cut this Line ſomewhere [p]); alſo let BF and DF be drawn. Becauſe $BFD + BCD = 2$ Right-angles [a] $= BAD + BCD$ [b], therefore muſt $BFD = BAD$ [c]: *Which is impoſſible* [d]. Therefore the Circumference of the Circle alſo paſſes thro' A. *Q. E. D.*

[p] Cor. to 3. 3.
[a] 13. 3.
[b] Hyp.
[c] Ax. 5. 1.
[d] 17. 1.

THEOREM XVI.

If two Right-lines AB, CD, terminated by the Periphery on both Sides, cut each other within the Circle, the Rectangle AP × BP, contained under the Parts of the one, will be equal to the Rectangle CP × DP contained under the Parts of the other.





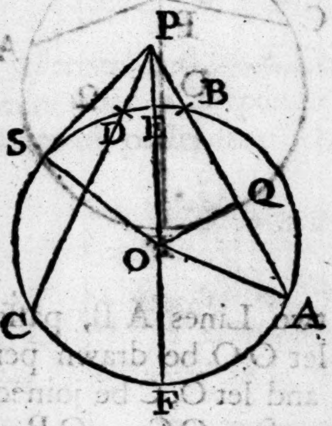
CASE 1. If one of the two Lines AB, paſſes thro' the Center O: Then let OQ be drawn perpendicular to the other CD, and let O, C be joined.

Becauſe $QD = QC$[a], therefore $QC - QP =$ [a] 1. 3.
DP[b]. But it is demonſtrated (*in* 8. 2.) that the [b] Ax. 5. 1.
Rectangle, under the Sum and Difference of the two Sides of a Triangle, is equal to the Rectangle under the whole Baſe and the Difference of its two Segments; therefore the Rectangle under $\overline{OC + OP}$ (AP) and $\overline{OC - OP}$ (BP) is equal to the Rectangle under CP and $\overline{CQ - PQ}$ (DP); that is $AP \times BP = CP \times DP$.

CASE 2. If neither of the two Lines paſs thro' the Center: Let the Diameter EPF be drawn; then, *by the preceding Caſe*, $AP \times BP = FP \times EP = CP \times DB$. *Q. E. D.*

THEOREM XVII.

If from any two Points A and C, in the Circumference of a Circle, two Right-lines AP, CP be drawn to paſs thro', and meet without, the Circle, the Rectangle AP×BP, contained under the Whole and the external Part of the one, will be equal to the Rectangle CP × DP contained under the Whole and the external Part of the other.



Through the Center O let the Right-line PF be drawn, meeting the Circumference in E and F; ſuppoſe OQ perpendicular to AP, and let A, O be joined.

Then, by 8. 2. the Rectangle, contained under $\overline{PO+OA}$ (PF) and $\overline{PO-OA}$ (PE), is = the Rectangle contained under AP and $\overline{PQ-AQ}$ (PB[b]); that is, PF × PE = AP × PB. After the very ſame Manner is PF × PE = CP × DP: Therefore AP × BP = CP × DP[c]. *Q. E. D.*

[b] 1. 3. and Ax. 5. 1.

[c] Ax. 1. 1.

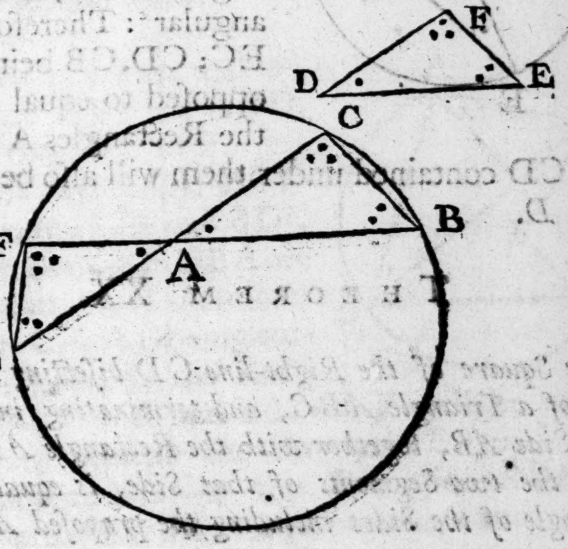
COROLLARY.

Hence, if PS be a Tangent at S, and the Radius OS be drawn; then PS^2[d] being = $\overline{PO+OS} \times \overline{PO-OS}$ (PF × PE), it follows that PC × PD = PS^2[e].

[d] Cor. to 7, 2.

THEOREM XVIII.

In equiangular Triangles ABC, DEF, the Rectangles, contained under the correſponding Sides taken alternately, are equal.



That is, if $A = D$, $B = E$ and $C = F$, then will $AB \times DF = AC \times DE$.

For, in BA produced let AF be taken $= DF$,[a] and let EFCB be the Circumference of a Circle paſſing thro' the three Points B, C, F[a], and meeting CA produced in E[b]; alſo let E, F be joined.

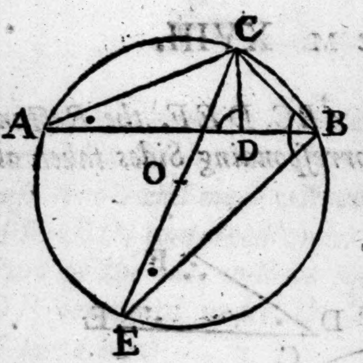
Becauſe the Angle $AEF = B$[c] $= DEF$[d], and $EAF = BAC$[e], alſo $AF = DF$[d], thence is $AE = DE$[f]; and therefore $AC \times DE = AC \times AE$[g], $= AB \times AF$[h] $= AB \times DF$[g]. *Q. E. D.*

[a] 14. 3.
[b] Cor. to 3. 3.
[c] Cor. to 9. 3.
[d] Hyp.
[e] 5. 1.
[f] 20. 1.
[g] 1. 2.
[h] 16. 3.

THEOREM XIX.

The Rectangle, under the two Sides AC, BC of a Triangle ABC, is equal to the Rectangle under the Perpendicular CD, and the Diameter CE of the circumſcribing Circle.

For,

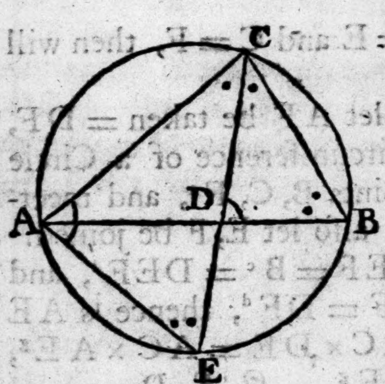


[a] Cor. to 9. 3.
[b] Hyp. and 11. 3.
[c] Cor. 2. to 10. 1.
[d] 18. 3.

For, B, E being joined, the Angles A and E will be equal [a], and ADC and EBC both Right-angles [b], and conſequently the Triangles ACD, ECB equiangular [c]: Therefore AC, EC; CD, CB being Sides oppoſed to equal Angles, the Rectangles AC × CB, EC × CD contained under them will alſo be equal [d]. *Q. E. D.*

THEOREM XX.

The Square of the Right-line CD biſecting any Angle C of a Triangle ABC, and terminating in the oppoſite Side AB, together with the Rectangle AD×BD under the two Segments of that Side, is equal to the Rectangle of the Sides including the propoſed Angle.



[a] 14. 3.

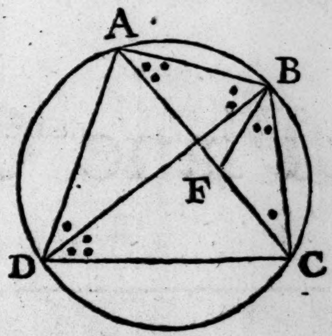
Let ACBE be the Circumference of a Circle ACBE paſſing thro' the three angular Points [a], and let CD, produced, meet the ſaid Circumference in E; alſo let the Subtenſe AE be drawn.

[b] Cor. to 9. 3.
[c] Cor. 2. to 10. 1.
[d] 18. 3.
[e] 4. 2.
[f] 16. 3. and Ax. 4. 1.

The Angles E and B, ſtanding upon the ſame Segment, are equal [b], and ACE is equal DCB (by Hypotheſis), therefore the Triangles AEC, DCB are equiangular [c], whereof AC, CD; CE, CB are correſponding Sides: Therefore $AC \times CB = CD \times CE$ [d] $= CD \times \overline{CD + DE} = CD^2 + CD \times DE$ [e] $= CD^2 + AD \times DB$. *Q. E. D.*

THEOREM XXI.

The Rectangle of the two Diagonals AC, BD of any Quadrilateral ABCD inſcribed in a Circle, is equal to the Sum of the two Rectangles $AB \times DC$, $AD \times BC$, under the oppoſite Sides.



Let BF be drawn, making the Angle $CBF = ABD$, and meeting AC in F.

Becauſe the Angle $BCF = ADB$[a], and $CBF = ABD$[b], the Triangles CBF, DBA are equiangular[c], and therefore, BC, BD; CF, AD being correſponding Sides, the Rectangles $BC \times AD$ and $BD \times CF$ will be equal[d]. Moreover, becauſe the Angle $ABF = CBD$[e], and $BAF = BDC$[a], the Triangles ABF and BDC are, likewiſe, equiangular, and conſequently, AB, BD; AF, DC being correſponding Sides, $AB \times DC = BD \times AF$[d]; to which let the equal Quantities $BC \times AD$ and $BD \times CF$ be reſpectively added, and then $AB \times DC + BC \times AD$[f] $= BD \times AF + BD \times CF$[f] $= BD \times AC$[g]. *Q. E. D.*

[a] Cor. to 9. 3.
[b] Conſtr.
[c] Cor. 2. to 10.. 1.
[d] 18. 3.
[e] Ax. 4. 1.
[f] 4. 2. and Ax. 4. 1.
[g] 4. 2.

The End of the THIRD BOOK.

ELEMENTS OF Plane Geometry.

BOOK IV.

DEFINITIONS.

1. RATIO is the Proportion which one Magnitude bears to another Magnitude of the ſame Kind, with reſpect to Quantity.

The Meaſure, or Quantity of a Ratio is conceived by conſidering what Part, or Parts the Magnitude referr'd, called the Antecedent, is of the other, to which it is referred, call'd the Conſequent.

2. Three Quantities, or Magnitudes A, B, C are A, B, C.
ſaid to be proportional, when the Ratio of the firſt 2. 4. 8.
A to the ſecond B, is the ſame as the Ratio of the
ſecond B, to the third C,

3. Four Quantities A, B, C, D are ſaid to be pro- A,B,C,D.
portional, when the Ratio of the firſt A to the 2.4.5.10.
ſecond B, is the ſame as the Ratio of the third C
to the fourth D.

To denote that four Quantities A, B, C, D are Proportional they are uſually wrote thus, $A : B :: C : D$; *and read thus, as A is to B, ſo is C to D. But when three Quantities A, B, C are proportional, the middle one is repeated, and they are wrote thus* $A : B :: B : C$.

4. Of three proportional Quantities, the middle one is ſaid to be a mean Proportional between the other two; and the laſt a third Proportional to the firſt and ſecond.

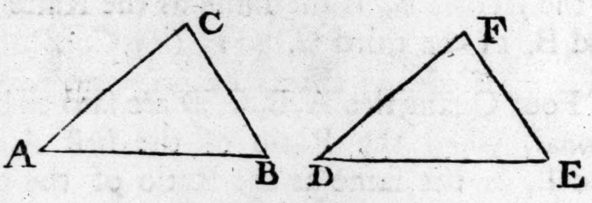
5. Of four proportional Quantities, the laſt is ſaid to be a fourth Proportional to the other three, taken in Order.

6. Quantities are ſaid to be continually propor- A,B,C,D,
tional (or in continual Proportion) when the firſt is E.
to the ſecond, as the ſecond to the third, as the 1.2.4.8.
third to the fourth, as the fourth to the fifth, and 16.
ſo on.

7. In a Series, or Rank of Quantities continually proportional, the Ratio of the firſt and third is ſaid to be duplicate to that of the firſt and ſecond, and the Ratio of the firſt and fourth, triplicate to that of the firſt and ſecond.

8. Similar (or like) right-lined Figures are ſuch which have all their Angles equal one to another reſpectively,

reſpectively, and alſo the Sides about the equal Angles proportional.



Thus if the Angle $A = D, B = E, C = F$; alſo $AC : AB :: DF : DE, BA : BC :: ED : EF$, *&c.* then the Figures ABC, DEF are alike.

AXIOMS.

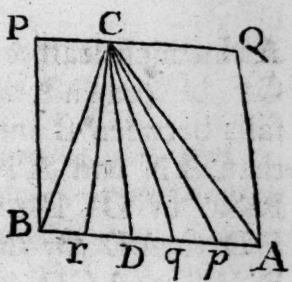
1. Quantities having the ſame Ratio to one and the ſame Quantity, or to equal Quantities, are equal among themſelves.
2. Equal Quantities, have to one and the ſame Quantity, the ſame Ratio.
3. Quantities, to which one and the ſame Quantity has the ſame Ratio, are equal.
4. If two Quantities be referred to a third, that is the greateſt which has the greateſt Ratio.
5. Ratios, equal to one and the ſame Ratio, are alſo equal one to the other.

Hence, if $A : B :: C : D$, *and* $C : D :: E : F$; *or, which is the ſame, if the Ratio of A to B = the Ratio of C to D = the Ratio of E to F, then will the Ratio of A to B = the Ratio of E to F, or* $A : B :: E : F$. *Moreover, when in two Ranks of proportional Quantities, as* $A : B :: C : D$ *and* $A : B :: C : E$, *the three firſt Terms are the ſame in both, the other two Terms D and E will be equal, by virtue of this, and the third Axiom, becauſe C has one and the ſame Ratio to them both, viz. that of A to B.*

Note. Whenever, in any Demonſtration, you meet with ſeveral proportional Quantities, connected continually by the Sign ∷ (like theſe, A : B ∷ C : D ∷ E : F ∷ G : H) the Concluſion to be drawn, is always from the firſt and laſt of the two equal Ratios.

THEOREM I.

Triangles ACD, BCD and Parallelograms ADCQ, BDCP, having the ſame Altitude, are to one another in the ſame Ratio as their Baſes AD and BD.



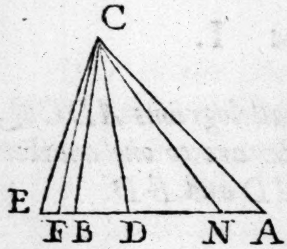
Let the Baſe AD be to the Baſe BD in the Ratio of any one Number M(3) to any other Number N (2,) or, which is the ſame, let AD contain M(3) ſuch equal Parts whereof BD contains N(2.) Then the Triangles ACp, pCq, BCr, *&c.* made by drawing from the Points of Diviſion to the Vertex C, will likewiſe be all equal among themſelves [c]. And the Triangle ACD will be to the Triangle BCD, as the Number of equal Parts in the former to the Number of equal Parts in the latter, or as the Number of Parts in AD to the Number of Parts in BD, that is, as AD to BD. Whence alſo, the Parallelograms ADCQ, BDCP, being the Doubles of their reſpective Triangles [c] are likewiſe as their Baſes AD and BD. *Q. E. D.*

[c] Cor. 2. to 2. 2.

SCHOLIUM.

SCHOLIUM.

If the Baſes AD and BD are incommenſurable to each other, the Ratio of the Triangles cannot be other than that of their Baſes.



For, if poſſible, let the Triangle BCD be to the Triangle ACD, not as BD to AD, but as ſome other Line DE (greater than BD) is to AD.

Let AN be a Part or Meaſure of AD, leſs than BE, and let DF be that Multiple of AN which leaſt exceeds DB; alſo, let CE and CF be drawn [b]. It is manifeſt that the Point F falls between B and E, becauſe *(by Hyp.)* BF is leſs than AN and AN leſs than BE. Moreover, the Ratio of DCF to ADC is the ſame as that of FD to AD *(by the precedent.)* But the Ratio of BCD to ACD (or of ED to AD, [c]) is $\sqsubset$ the Ratio of FD to AD [d], or of FCD to ACD; and conſequently BCD $\sqsubset$ FCD [e]: *Which is impoſſible* [f]. By the ſame Argument it will appear, that the Triangle BCD cannot be to the Triangle ACD, as a Line, leſs than BD, is to AD. Therefore BCD : ACD :: BD : AD. *Which was to be proved* *.

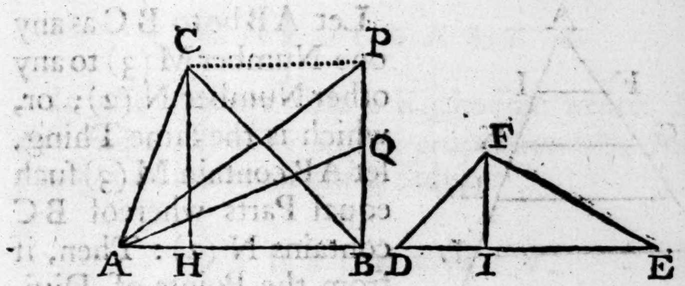
[b] Ax. 2. of 1.
[c] Hyp.
[d] Ax. 4. of 4.
[e] Ax. 4. of 4.
[f] Ax. 2. of 1.

* Note. *If this Scholium ſhould appear difficult to the Beginner, it may not be amiſs if he entirely omits it; ſince it is only put down for the Sake of thoſe who may be ſcrupulous about the Buſineſs of Incommenſurables, to whom it may not be improper to obſerve, that nothing more is taken for granted here, than what immediately follows from the Poſtulatum in the 10th Book of Euclid's Elements.*

THEOREM

THEOREM II.

Triangles ABC, DEF ſtanding upon equal Baſes AB, DE are to one another as their Altitudes CH and FI.



Let BP be perpendicular to AB and equal to CH, in which let there be taken $BQ = FI$, and let AP and AQ be drawn.

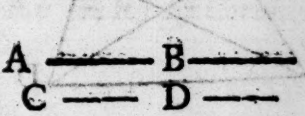
The Triangle ABP is $= ABC$[a], and $ABQ =$ DEF[a]; but ABP (ABC) : ABQ (DEF) ::[c] BP (HC) : BQ (FI.) *Which was to be demonſtrated.*

[a] Cor. 2. to 2. 2.
[c] 1. 4.

THEOREM III.

If four Lines A, B, C, D are proportional, the Rectangle $A \times D$ contained under the two Extremes is equal to the Rectangle $B \times C$ contained under the two Means. And, if the Rectangle $A \times D$ under the two Extremes be equal to the Rectangle $B \times C$ under the two Means, then are the four Lines proportional.

1. Hyp. $A \times D : B \times D$ $(A : B)$[a] :: $C : D$[b] :: $B \times C : B \times D$[a]; whence $A \times D = B \times C$[c].

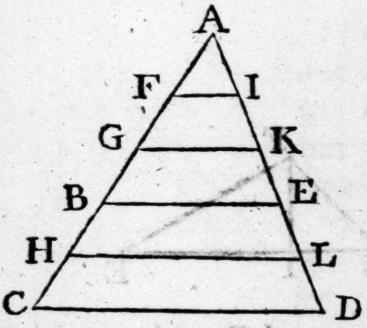


2. Hyp. $A : B :: A \times D : B \times D$[a] :: $B \times C : B \times D$[d] :: $C : D$[a]. *Which was to be demonſtrated.*

[a] 1. 4.
[b] Hyp.
[c] Ax. 1. of 4.
[d] Ax. 2. of 4.

THEOREM IV.

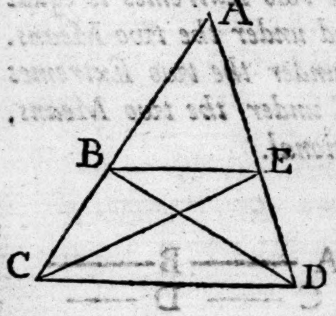
If a Right-line BE be drawn parallel to one Side CD of a Triangle ACD, it will cut the other two Sides proportionally, viz. *AB*:*BC*::*AE*:*ED*.



Let AB be to BC as any one Number M (3) to any other Number N (2); or, which is the ſame Thing, let AB contain M (3) ſuch equal Parts whereof BC contains N (2): Then, if from the Points of Diviſion Right-lines FI, GK, *&c.* be drawn parallel to the Side CD, they will alſo divide AE and DE into the ſame Number of equal Parts [b]. And therefore AE will be to DE, as the Number of Parts in AE to the Number of Parts in DE, or as the Number of Parts in AB to the Number of Parts in BC, or laſtly, as AB to BC.

[b] Cor. 1. to 27. 1.

Otherwiſe.



Draw CE and BD. Then, the Triangles BEC, EBD, upon the ſame Baſe BE and between the ſame Parallels BE and CD, being equal [c], it will be AB: BC :: (BEA : BEC, [d] :: BEA : BED [c]) :: AE : ED [d]. *Which was to be demonſtrated.*

[c] Cor. 1. to 2. 2.
[d] 1. 4.
[e] Ax, 2. of 4.

COROLLARY I.

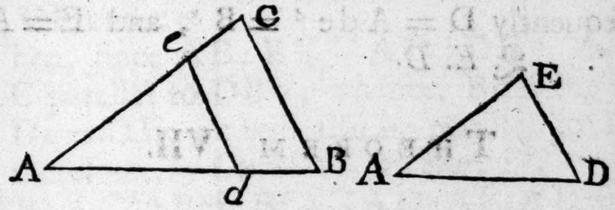
Hence, alſo, AC : AB :: AD : AE : For, AC : AB :: AEC : AEB [d] :: ABD : AEB [e] :: AD : AE [d]

COROLLARY 2.

It alſo appears that a Right-line which divides two Sides of a Triangle proportionally, muſt be parallel to the remaining Side.

THEOREM V.

The correſponding Sides of equiangular Triangles ABC, ADE are proportional (AB : AD :: AC : AE).



In AB take Ad = AD, and in AC take Ae = AE, and draw de. The Triangles Ade, ADE, having two Sides and the contained Angle, in the one, equal to two Sides and the contained Angle, in the other [p], will alſo have Ade = ADE [q] = ABC [r]; and ſo, de being parallel to BC [s], it will be AB : Ad (AD) :: AC : Ae (AE) *by Corol.* 1. *to the precedent.* *Q. E. D.*

Otherwiſe.

Becauſe AB × AE = AD × AC [a], therefore AB : AD :: AC : AE [b]. *Q. E. D.*

[p] Hyp.

[q] Ax. 9. of 1.

[r] Hyp.

[s] Cor. to 9. 1.

[a] 18. 3.

[b] 3. 4.

COROLLARY.

[c] Def. 8. of 4.

Hence, equiangular Triangles are ſimilar to each other[c].

THEOREM VI.

If two Triangles ABC, ADE (ſee the preceding Figure) *have one Angle BAC equal to one Angle DAE, and the Sides about the equal Angles proportional*; viz. *AB : AD ∷ AC : AE, then are thoſe Triangles equiangular.*

[a] Cor. 1. to 4. 4.
[b] Hyp.
[c] Ax. 5. and 3. 4.
[d] Ax. 9. of 1.
[e] Cor. 1. to 8. 1.

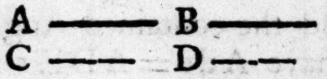
In AB take Ad = AD, and let de be parallel to BC, meeting AC in e.

Becauſe AB : Ad (AD) ∷ AC : Ae[a], and AB : AD ∷ AC : AE[b], therefore is Ae = AE[c], and conſequently D = Ade[d] = B[e], and E = Aed[d] = C[e]. *Q. E. D.*

THEOREM VII.

If four Lines A, B, C, D be proportional, viz. *A : B ∷ C : D, they will be ſo alternately,* viz. *A : C ∷ B : D.*

[a] 3. 4.
[b] 1. 4.
[c] Ax. 2. 4.



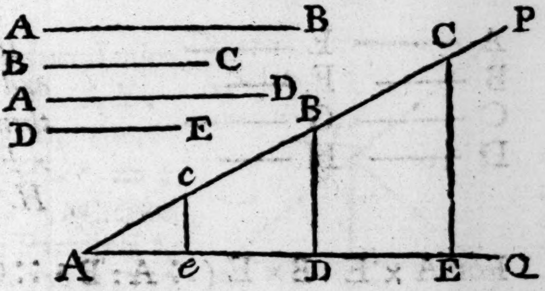
For, ſince A : B ∷ C : D, therefore is A × D = B × C[a]; but A : C (∷ A × D : C × D[b] ∷ B × C : C × D[c]) ∷ B : D[a]. *Q. E. D.*

THEOREM

THEOREM VIII.

If four Lines AB, BC, AD, DE be proportional, then,

Compoundedly, $AB : AB + BC :: AD : AD + DE$.
Dividedly, $AB : AB - BC :: AD : AD - DE$.
Mixtly, $AB + BC : AB - BC :: AD + DE : AD - DE$.



For, from any Point A let two indefinite Right-lines AP, AQ be drawn, in which take AB = AB, BC = BC, *&c.* alſo take Bc = BC, De = DE, and draw BD, CE and ce.

Then, ſince AB : BC :: AD : DE [a], therefore is EC parallel to DB [b]; whence, Bc being = BC and De = DE, ec will alſo be parallel to DB [c]; and therefore, *by Coroll.* 1. *to Theor. IV.*

[a] Hyp.
[b] Cor. 2. to 4. 4.
[c] Cor. 2. to 27. 1.

$AB : AC (AB + BC) :: AD : AE (AD + DE)$.
$AB : Ac (AB - BC) :: AD : Ae (AD - DE)$.
$AC (AB + BC) : Ac (AB - BC) :: AE (AD + DE) : Ae (AD - DE.)$ *Which was to be demonſtrated.*

SCHOLIUM.

What is demonſtrated in the two laſt Theorems concerning the Ratios of Lines, muſt, it is manifeſt, hold equally true in the Ratios of any other Kind of Magnitudes or Quantities: Becauſe, let the Quantities compared be of what Kind they will, Right-lines may be aſſumed having the ſame Ratio; and, if the Ratios upon which the Concluſions depend be the ſame, the Concluſions themſelves muſt conſequently be the ſame, whether thoſe Ratios are to be added, ſubſtracted, multiplied or divided, *&c.*

THEOREM IX.

The Rectangles under proportional Lines, are themſelves proportional; that is,

If $A : B :: C : D$ *and* $E : F :: G : H$; *then will* $A \times E : B \times F :: C \times G : D \times H$.

For $A \times E : B \times E$ ($:: A : B$[a] $:: C : D$[b]) $:: C \times G : D \times G$[a].

[a] 1. 4. $B \times E : B \times F$ ($:: E : F$[a] $:: G : H$[b]) $:: D \times G : D \times H$[a].

[b] Hyp. Whence, alternately, $A \times E : C \times G :: B \times E : D \times G$[c].

[c] 7. 4. $B \times E : D \times G :: B \times F : D \times H$[c].

Therefore, by Equality, $A \times E : C \times G :: B \times F : D \times H$[e], and conſequently $A \times E : B \times F :: C \times G : D \times H$[e]. *Which was to be demonſtrated.*

[e] Ax. 5. 4.

COROLLARY 1.

Hence, the Squares upon four proportional Lines are alſo proportional.

COROLLARY 2.

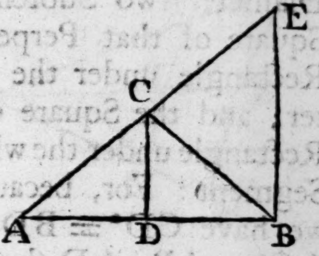
Hence, alſo, if four Squares be proportional, ($A^2 : B^2 :: C^2 : D^2$) their Sides A, B, C, D will be pro-

proportional; For, let $A : B :: C : E$, then, ſince $A^2 : B^2 :: C^2 : E^2$ (by the preceding Coroll.) therefore $E^2 = D^2$ [e], and conſequently $E = D$ [f].

[e] Ax. 3. and 5. 4.
[f] Cor. 2. to 5. 2.

THEOREM X.

A Right-line CD biſecting any Angle ACB of a Triangle ABC, and falling upon the oppoſite Side AB, divides that Side into Segments AD, BD, having the ſame Ratio, as the Sides AC, CB including that Angle.

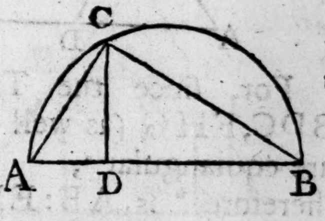


For in AC produced take $CE = BC$ and join B, E: Since $CE = CB$, therefore is $E = EBC$ [a] $= \frac{1}{2} ACB$ [b] $= ACD$ [c]; whence CD and EB are Parallels [d], and therefore $AD : BD :: AC : (BC)\, CE$ [e]. *Q. E. D.*

[a] 6. 1.
[b] 10. 1.
[c] Hyp.
[d] Cor. to 9. 1.
[e] 4. 4.

THEOREM XI.

In a right-angled Triangle ABC, a Perpendicular CD let fall from the Right-angle C, upon the Hypothenuſe AB, will be a mean Proportional between the two Segments AD, BD of the Hypothenuſe: And each of the Sides including the Right-angle will be a mean Proportional between its adjacent Segment and the whole Hypothenuſe.



For, ſince the Angle $BDC = BCA$ [a], and B common, the Triangles BDC, BCA are equiangular [b]. After the ſame Manner are ADC, ABC equiangular, and conſequently ACD equiangular to BCD [c]: There-

[a] Ax. 7.
[b] Cor. 2. to 10. 1.
[c] Ax. 1. 4.

Therefore, $BD : CD :: CD : AD$
by Theor. V. $AB : BC :: BC : BD$
$AB : AC :: AC : AD$.

Which was to be demonſtrated.

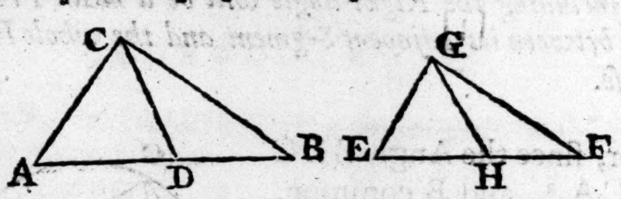
COROLLARY.

[d] 11. 3.

Becauſe the Angle in a Semi-circle is a Right-angle [d], it follows that, if from any Point C, in the Periphery of a Semicircle ACB, a Perpendicular CD be let fall upon the Diameter AB, and from the ſame Point C, to the Extremities of that Diameter two Subtenſes CA, CB be drawn; the Square of that Perpendicular will be equal to a Rectangle under the two Segments of the Diameter; and the Square of each Subtenſe equal to a Rectangle under the whole Diameter and its adjacent Segment: For, becauſe of the above Proportions, we have $CD^2 = BD \times AD$, $BC^2 = AB \times BD$ and $AC^2 = AB \times AD$ (by Theorem III.)

THEOREM XII.

If, in ſimilar Triangles ABC, EFG, from any two equal Angles ACB, EGF two Right-lines CD, GH be drawn making equal Angles with the homologous Sides CB, GF; thoſe Right-lines will have the ſame Ratio as the Sides AB, EF on which they fall, and will alſo divide thoſe Sides proportionally.



[a] Hyp.
[b] 5. 4 and Ax. 5.

For, ſince the Triangles ADC, EHG, and BDC, FHG (as well as the *Wholes* ABC, EFG) are equiangular [a],
therefore [b] is $AB : EF (:: AC : EG) :: CD : GH$
$AD : EH (:: DC : HG) :: BD : FH$.

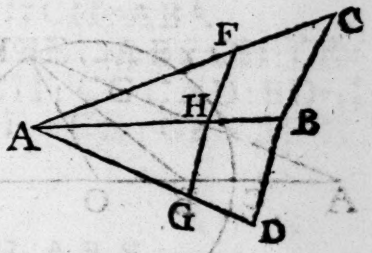
Q. E. D.

THEOREM

THEOREM XIII.

If two Triangles ABC, ABD have one Side AB common to both, and from any Point H in that Side there be drawn two Lines HF, HG reſpectively parallel to two of the remaining Sides BC, BD ſo as to terminate in the other two Sides AC, AD; thoſe Lines HF, HG will have the ſame Ratio as the Sides BC, BD, to which they are parallel.

Eor AB : AH :: BC : HF[a], and AB : AH :: BD : HG[a]; therefore, by Equality, BC : HF :: BD : HG; whence, alternately, BC : BD :: HF, HG[b]. *Q. E. D.*



[a] 5. and 7. 4.
[b] 7. 4.

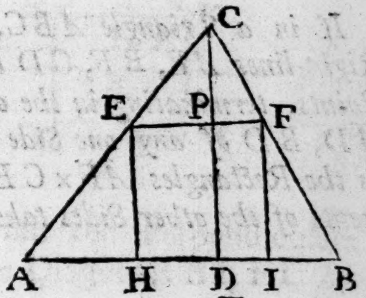
COROLLARY.

Hence, if BC = BD, then alſo will HF = HG.

THEOREM XIV.

The Side EF (or EH) of a Square EFIH inſcribed in a Triangle ABC, is to the Baſe of the Triangle AB, as the Perpendicular CD to the Sum of the Baſe and Perpendicular AB+CD.

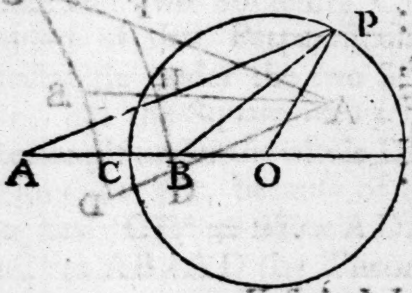
For EF (PD) : CP :: AB : CD[a], therefore, compoundedly EF : CD :: AB : AB+CD[b], and conſequently EF : AB :: CD : AB+CD[c]. *Q. E. D.*



[a] 12. 4.
[b] 8. 4.
[c] 7. 4.

THEOREM XV.

If a Rigt-line AB be unequally divided in C, and in it produced there be taken CO to AC, as BC to AC — CB, and upon O, as a Center, at the Diſtance of OC, a Circle CPD be deſcribed, and two Lines AP, BP be drawn from A and B to meet any where in the Periphery thereof; thoſe Lines AP, BP will be (every where) in the conſtant Ratio of AC to BC.



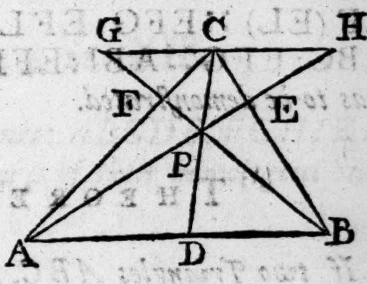
Draw the Radius OP. Since CO : AC :: BC : AC — CB[a], therefore, by Compoſition, CO : AO :: BC : AC[b]; whence, alternately, CO : BC :: AO : AC[c], and therefore, dividedly, CO : BO :: AO : CO[b], or PO : BO :: AO : PO[d]. Wherefore, ſeeing the Sides about the common Angle O, are proportional, the Triangles BOP, POA are equiangular[e], and therefore their other Sides alſo proportional, that is PO (CO) : AO :: PB : AP[f]; whence, by Equality, BP : AP :: BC : AC. *Q. E. D.*

THEOREM XVI.

If in a Triangle ABC, thro' any Point P, three Right-lines AE, BF, CD be drawn from the angular Points, terminating in the oppoſite Sides, the Segments AD, BD of any one Side AB, will be to each other as the Rectangles AF × CE, BE × CF under the Segments of the other Sides taken alternately.

[a] Hyp.
[b] 8. 4.
[c] 7. 4.
[d] Ax. 2. and 3. 4.
[e] 6. 4.
[f] 5. 4.

Let GCH be parallel to AB, and let AE and BF be produced to meet it in H and G.



[a] 8. 1. and 5. 1.

It is manifeſt that the Triangles FBA, FCG; EAB, EHC; APB, GPH, are equiangular[a].

Therefore $AF : CF :: AB : CG$[b],
and $CE : BE :: CH : AB$[b].

Whence $AF \times CE : CF \times BE :: AB \times CH : CG \times AB$[c] $:: CH : CG$[d]; but $CH : CG :: AD : BD$[e]; therefore, by Equality, $AF \times CE : CF \times BE :: AD : BD$. *Q. E. D.*

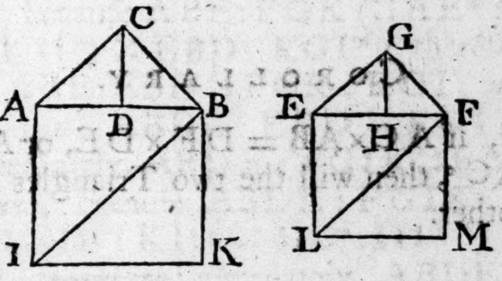
[b] 5. 4.
[c] 9. 4.
[d] 1. 4.
[e] 12. 4.
[f] 3. 4.

COROLLARY.

Hence, if $AD = BD$, then alſo will $AF \times CE = CF \times BE$, and therefore $AF : CF :: BE : CE$[f].

THEOREM XVII.

Equiangular Triangles ABC, EFG are to one another as the Squares AK, EM of their homologous Sides.



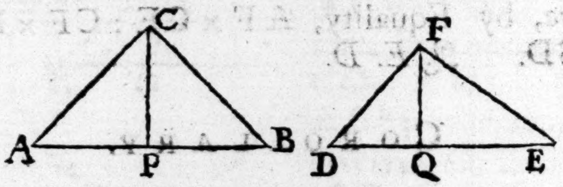
Upon AB and EF let fall the Perpendiculars CD and GH, and let the Diagonals BI, FL be drawn.

Becauſe

[a] 1. 4. Becauſe ABC : ABI :: CD : AI (AB)[a] :: GH :[b]
[b] 12. 4. EF (EL) :: EFG : EFL[a]; therefore, alternately,
[c] 7. 4. ABC : EFG :: ABI : EFL[c] :: AK : EM[d]. *Which*
[d] Cor. 2. *was to be demonſtrated.*
to 2. 2.

THEOREM XVIII.

If two Triangles ABC, DEF have one Angle A, in the one, equal to one Angle D, in the other, thoſe Triangles will be to each other in the Ratio of the Rectangles AB × AC, DE × DF contained under the Sides including the equal Angles.



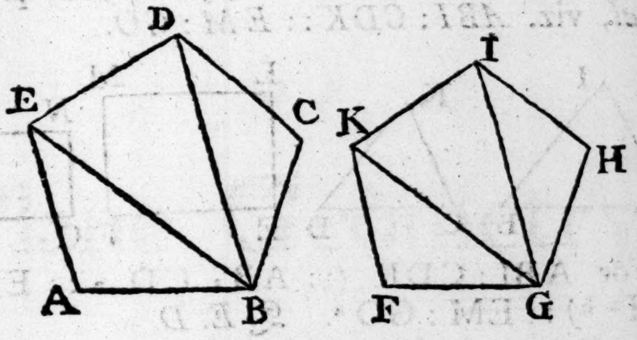
Upon AB and DE let fall the Perpendiculars
CP and FQ. The Triangles ACP, DFQ, are
[a] Cor. 2. equiangular[a], therefore AC : CP :: DF : FQ[b].
to 10. 1. But AC × AB : CP × AB (:: AC : CP[c] :: DF :
[b] 5. 4. FQ) :: DF × DE : FQ × DE[c]; therefore, alter-
[c] 1. 4.
[d] Cor. 2. nately, AC × AB : DF × DE :: CP × AB : FQ
to 2. 2. × DE :: the Triangle ABC : Triangle DEF[d].
[e] 3. 4. *Q. E. D.*

COROLLARY.

Hence, if AC × AB = DF × DE, or AB : DE :: DF : AC[e], then will the two Triangles be equal to each other.

THEOREM XIX.

Similar right-lined Figures ABGDE, FGHIK are to one another as the Squares of their homologous Sides.



Draw BE, BD, GK, GI.

Becauſe A = F and AB : AE :: FG : FK [a], the Triangles BAE, GFK are equiangular [b]; therefore, if from AED = FKI [a], there be taken AEB = FKG, the Remainders BED, GKI will alſo be equal [c]. Wherefore, ſince ED : KI (:: EA : KF [a]) :: EB : KG, the Triangles EBD, KGI are likewiſe equiangular [b]. And by reaſoning in the ſame Manner it will appear that DBC, IGH are alſo equiangular.

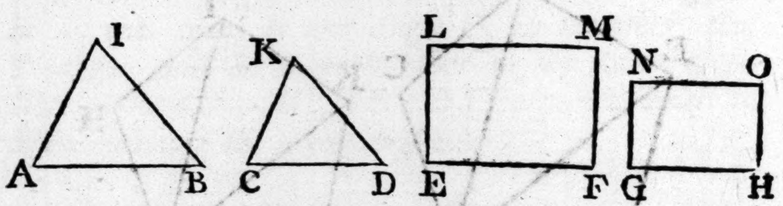
Now, becauſe ABE : FGK (:: $AE^2 : FK^2$ [d] :: $ED^2 : KI^2$ [e]) :: EBD : KGI [d]; therefore, alternately, ABE : EBD :: FGK : KGI; whence, by Compoſition, ABDE : EBD :: FGIK : KGI, and, alternately, ABDE : FGIK :: EBD : KGI. Moreover, becauſe ABDE : FGIK (:: EBD : GKI :: $ED^2 : KI^2$ [d] :: $DC^2 : IH^2$ [e]) :: DBC : IGH [d], therefore, alternately, ABDE : DBC :: FGIK : IGH; whence, again by Compoſition and Alternation, ABCDE : FGHIK :: DBC : IGK (::[d] $DC^2 : IH^2$ ::[e] $ED^2 : KI^2$) :: $AE^2 : FK^2$, &c. *Which was to be demonſtrated.*

THEOREM

[a] Def. 8. 4.
[b] 6. 4.
[c] Ax. 6. of 1.
[d] 17. 4.
[e] Cor. 1. to 9. 4.

THEOREM XX.

If four Right-lines AB, CD, EF, GH be proportional, the right-lined Figures deſcribed upon them, being like, and in like Sort ſituate, ſhall alſo be proportional, viz. *ABI* : *CDK* :: *EM* : *GO*.



a 17. 4.
b Cor. 1. to 9. 4.

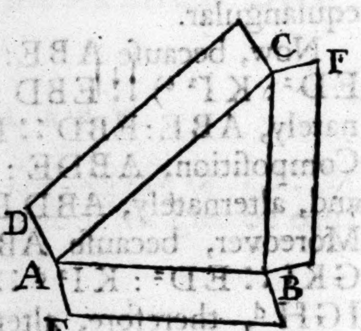
For ABI : CDK (:: AB_2 CD [a] :: EF^2 : GH^2 [b]) :: EM : GO [a]. *Q. E. D.*

THEOREM XXI.

If upon the three Sides of a right-angled Triangle ABC as many right-lined Figures CD, BE, BF, like and alike ſituate, be deſcribed, that CD upon the Hypothenuſe AC, will be equal to both the other two BE, BF taken together.

a 17. 4.
b 7. 2.

For $AB^2 : BC^2$ :: BE : BF [a], therefore, compoundedly, $AB^2 : AB^2 + BC^2$ ($= AC^2$ [b]) :: BE : BE + BF; but $AB^2 : AC^2$ :: BE : CD [a]; therefore, the three firſt Terms of the two laſt Proportions being the ſame, the remaining Terms muſt be equal, that is, BE + BF = CD [c]. *Which was to be demonſtrated.*



c Ax. 3. and 5. 4.

The End of the FOURTH BOOK.

ELEMENTS

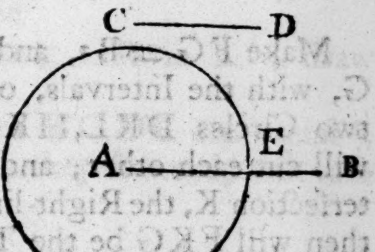
OF

Plane Geometry

BOOK V.

PROBLEM I.

FROM the greater AB of two unequal Right-lines AB, CD to cut off, or take away a Part AE equal to the leſſer CD.

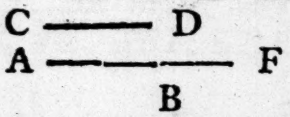


From the Center A, with the Radus CD, let a Circle be deſcribed [a] cutting AB in E [b]; *and the Thing is done.*

[a] Poſt. 3.
[b] Cor. to 3. 3.

PROBLEM II.

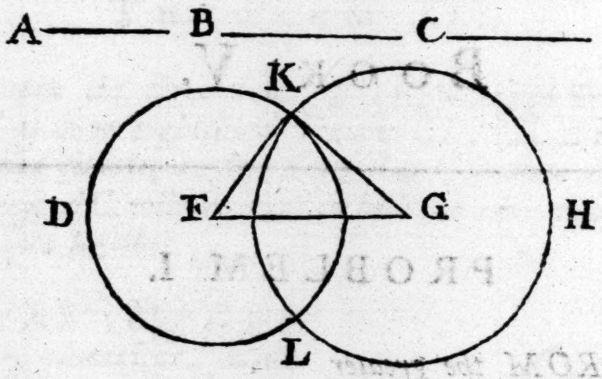
At a given Point A, to make a Right-line AB equal to a given Right-line CD.



[a] Poſt. 1. and 2. [b] 1. 5. Draw the infinite Right-line AF[a]; from which take away AB = CD[b]; *and the Thing is done.*

PROBLEM III.

To deſcribe a Triangle whoſe three Sides ſhall be equal to three given Lines A, B, C; provided any two of them, taken together, be greater than the third.



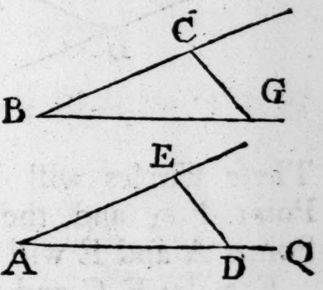
[a] 2. 5. [b] Poſt. 3. Make FG = B[a], and from the Centers F and G, with the Intervals, or Diſtances A and C, let two Circles DKL, HKL be deſcribed[b]; which will cut each other; and, if from the Point of Interſection K, the Right-lines KF and KG be drawn, then will FKG be the Triangle required.

[c] Ax. 6. 1. [d] Hyp. [e] 8. 3. [f] Def. 29. 1. For *(by Hyp.)* $B + C \sqsubset A$, therefore $B\,(FG) \sqsubset A - C$[c]; but $B\,(FG) \sqsupset A + C$[d]; therefore the two Circles cut each other[e]: Conſequently $FK = A$, $FG = B$, and $GK = C$[f]. *Q. E. D.*

PROBLEM

PROBLEM IV.

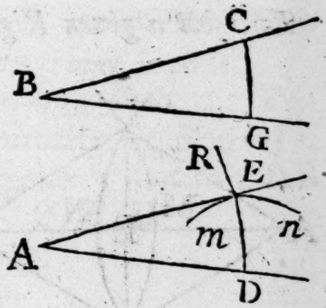
At a given Point A, in a given Right-line AQ, to make an Angle A equal to a given Angle B.



Draw any Right-line GC cutting the two Lines including the given Angle G and C; take AD = BG[b], and upon AD (*by the precedent*) let a Triangle be conſtituted, whereof the Side DE = GC and the Side AE = BC; then alſo will the Angle A = Angle B[c]. *Which was to be done.*

b 1. 5.
c 19. 1.

Otherwiſe.



Upon the Centers A and B, at any Diſtance AD (BG) let two Circles GC, DR be deſcribed interſecting the given Lines in D, C and G[d]; alſo upon D as a Center, with a Radius equal to the Diſtance of the Points G and C[e], let another Circle mEn be deſcribed, cutting DR in E[f]; draw AE, *and the Thing is done.*

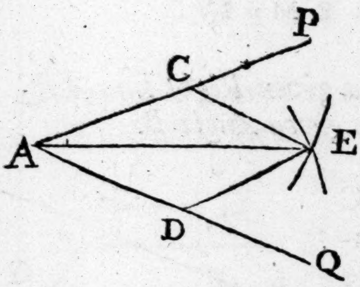
d Cor. to 3. 3.
e Poſt. 3.
f 8. 3.

For, ſince the Subtenſes GC, DE are equal[g], the Angles B and A which they ſubtend, will alſo be equal[h].

g Conſtr.
h 19. 1.

PROBLEM V.

To biſect, or divide into two equal Parts, any given right-lined Angle PAQ.

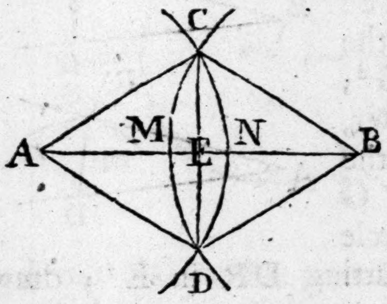


In the Lines containing the given Angle,
take AC = AD[a], and
a 1. 5. upon the Centers C and
D, with any Radius or
Interval greater than
half the Diſtance of the
Points C and D, let two
b Poſt. 3. Circles be deſcribed[b]:
c 8. 3. Theſe Circles will interſect each other in ſome
Point E[c]; and the Right-line AE joining the
Points A and E will biſect the given Angle PAQ.

For, let E, C and E, D be joined; then becauſe
d Conſtr. AC = AD[d], CE = DE[d] and AE common,
e 19. 1. therefore is CAE = DAE[e].

PROBLEM VI.

To biſect a given Right-line AB.



From the Centers A
and B, with any Radius
greater than half AB,
deſcribe two Circles D
NC, DMC[a], cutting
a Poſt. 3. each other in the Points
b 8. 3. C and D[b]; draw CD,
interſecting AB in E,
and the Thing is done.

For, if AC, AD, BC and BD be drawn, the Triangles ACD, BCD will be mutually equilateral[c];
c Conſtr. and conſequently the Angle ACE = the Angle
d 19. 1. BCE[d]: Therefore, the Triangles ACE, BCE,
e Ax. 9. 1. having AC = BC[c], CE common, and the Angle
ACE = BCE, will alſo have the Baſe AE = the
Baſe BE[e]. *Q. E. D.*

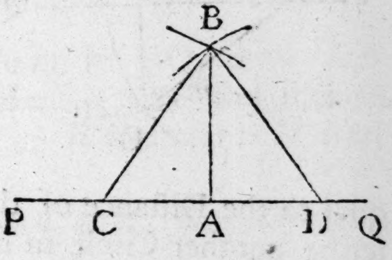
COROLLARY.

Hence, it is manifeſt, that CD not only biſects AB, but is alſo perpendicular to it [f].

[f] Ax. 9. and Def. 6. of 1.

PROBLEM VII.

At a given Point A, in a given Right-line PQ, to erect a Perpendicular.

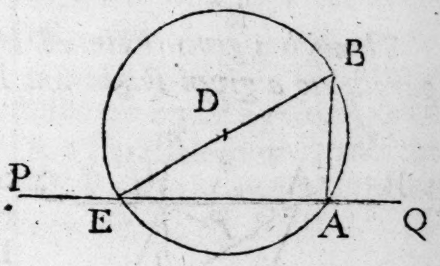


In the given Line, on either Side of A, take any two equal Diſtances AC and AD [a], and from the Centers C and D, with any Interval greater than AC (or AD) let two equal Circles be deſcribed [b], interſecting each other in B [c]; join A, B, *and the Thing is done.*

For, let CB and DB be drawn; then the Triangles CAB, DAB being mutually equilateral [e], will alſo be mutually equiangular [f], and ſo CAB = DAB = a Right-angle [g]. *Q. E. D.*

[a] 1. 5.
[b] Poſt. 3.
[c] 8. 3.
[e] Conſtr.
[f] 19. 1.
[g] Def. 6. of 1.

Otherwiſe.



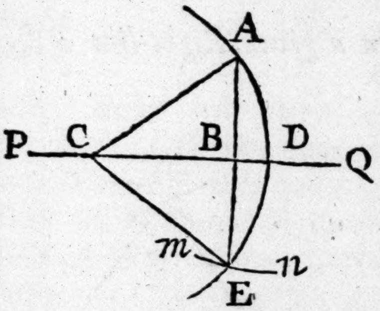
From any Point D, not in the Line PQ, with a Radius or Interval equal to the Diſtance of the two Points A and D, let a Circle be deſcribed [a], cutting PQ in A and E [b]; draw the Right-line EDB [c], meeting the Periphery in B [b]; join

[a] Poſt. 3.
[b] Cor. to 3. 3.
[c] Poſt. 1. and 2.

join A, B; then the Angle EAB, being in the
[d] 11. 3. Semi-circle EAB, is a Right-angle [d]. *Which was to be done.*

PROBLEM VIII.

From a given Point A, upon an infinite Right-line PQ, to let fall a Perpendicular.



From any Point C in the Line PQ, thro' the given Point A, let the Circumference of a Circle ADE be deſcribed [a], interſecting PQ in D [b]; and from the Center D, with a Radius or Interval equal to the Diſtance of the Points A and D, deſcribe another Circle mEn [a], cutting the former in E [c]; then draw ABE for the Perpendicular required.

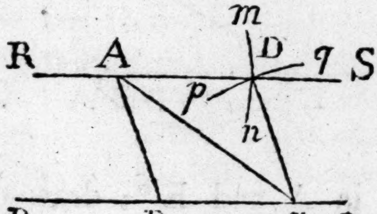
[a] Poſt. 3.
[b] Cor. to 3. 3.
[c] 8. 3.

For, let AC and CE be drawn; then the Subtenſes of the Angles DCA, DCE being equal, by Conſtruction, the Angles themſelves will be equal [d]; therefore, ſince CA = CE [e], and CB common, the Angle CBA is alſo = CBE [f], and conſequently AB perpendicular to PQ [g]. *Q. E. D.*

[d] 19. 1.
[e] Def. 29. of 1.
[f] Ax. 9. of 1.
[g] Def. 6. of 1.

PROBLEM IX.

Through a given Point A, to draw a Right-line RS parallel to a given Right-line PQ.



Take any Point B in the given Line, and make BC equal to the Diſtance of the Points A and B [a]; and from the Centers A and C, with the Radius BC, let

[a] 1. 5.

let two Circles be deſcribed [b]: Theſe Circles will cut each other in ſome Point D, and the Right-line RS drawn thro' D and A will be parallel to PQ.

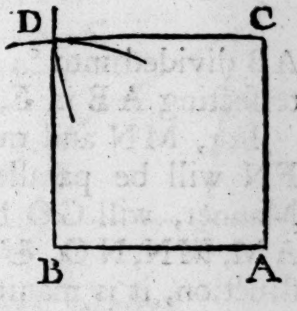
Let AB and AC be drawn [c]: It is plain that the two Circles will cut each other [d], becauſe the Sum of their Semi-diameters ($=AB+BC$[e]) is greater than AC [f]: Therefore, if AD and CD be alſo drawn, we ſhall have $AB=BC=CD=DA$[g], [h] and ſo RS will be parallel to PQ [h]. *Q. E. D.*

The ſame otherwiſe.

From A, to any Point in PQ, draw AB [a]; make the Angle $SAB=PBA$ [b]; and then AS will be parallel to PQ [c]. *Which was to be done.*

PROBLEM X.

Upon a given Right-line AB to deſcribe a Square ABCD.



Make AC perpendicular, and equal to AB [a], and from the Centers B, C, let two Circles, with the Radius AB or AC, be deſcribed [b], interſecting each other in D: From the Point of Interſection draw DB and DC, *and the Thing is done.*

For, all the four Sides are equal by Conſtruction; therefore the Figure is a Parallelogram [d], and therefore, the Angle A being a Right-angle [f], the other three are all Right-angles [g]. *Q. E. D.*

[b] Poſt. 3.
[c] Poſt. 1.
[d] 8. 3.
[e] Conſtr.
[f] 15. 1.
[g] Ax. 1.
[h] 24. 1.

[a] Poſt. 1.
[b] 4. 5.
[c] 9. 1.

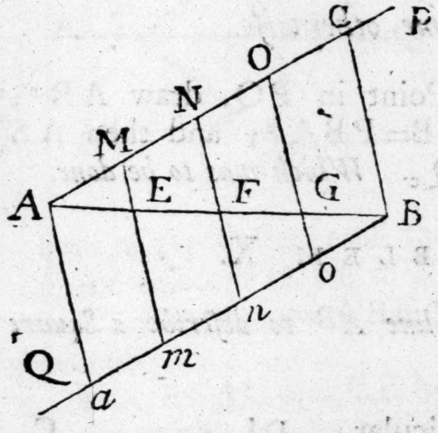
[a] 7. 5. and 1. 5.
[b] Poſt. 3.
[d] 24. 1.
[f] Conſtr.
[g] Cor. to 23. 1.

SCHOLIUM.

By the ſame Method a Rectangle may be deſcribed, the Sides thereof being given.

PROBLEM XI.

To divide a given Right-line AB into any propoſed Number of equal Parts.



a Poſt. 1. and 2.
b 9. 5.

Draw the infinite Right-line AP [a], making any Angle with AB; alſo draw BQ parallel to AP [b], in each of which Lines let there be taken as many equal Parts AM, MN, *&c.* Bo, on, *&c.* (of any Length at Pleaſure) as you would have AB divided into [c]; then draw Mm, Nn, *&c*, interſecting AB in E, F, *&c. and the Thing is done.*

c 1. 5.

d Conſtr.
e 25. 1.

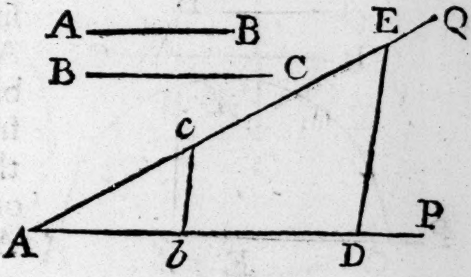
For, MN and mn being equal and parallel [d], FN will be parallel to EM [e]; and, in the ſame Manner, will GO be parallel to FN: Therefore, AM, MN, NO, *&c.* being all equal, by Conſtruction, it is manifeſt that AE, EF, FG, *&c.* are likewiſe equal [f]. *Q. E. D.*

f Cor. 1. to 27. 1.

PROBLEM XII.

To find a third Proportional to two given Right-lines AB, BC.

From

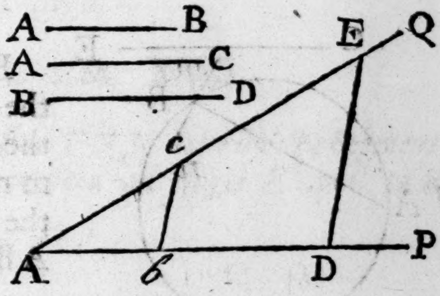


From any Point A, draw two infinite Right-lines A P, A Q [a], in which take A b = A B, A c = BC, and b D = BC[e]; draw b c [f], and parallel to b c, draw DE [g] cutting A Q in E; then cE will be the third Proportional required: For A b (AB): A c (BC) :: b D (BC) : c E [h]. *Which was to be done.*

[a] Poſt. 1. and 2.
[e] 1. 5.
[f] Poſt. 1.
[g] 9. 5.
[h] 4. 4.

PROBLEM XIII.

To find a fourth Proportional, to three given Right-lines AB, AC, B D.



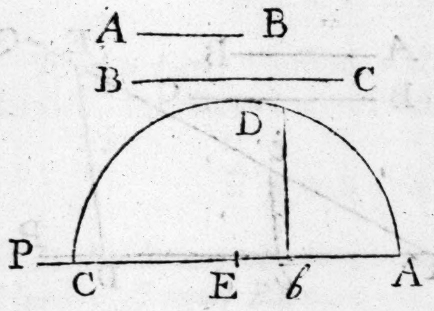
Having drawn A P and A Q, as in the preceding Problem, take therein A b = A B, A c = A C and b D = B D [a]; draw b c [e], and parallel to it, draw D E [f] interſecting A Q in E; then is c E *the fourth Proportional required.*

For A b (AB) : A c (AC) :: b D (BD) : c E [g]. *Which was to be done.*

[a] 1. 5.
[e] Poſt. 1
[f] 9. 5.
[g] 4. 4.

PROBLEM XIV.

To find a Mean-proportional between two given Right-lines AB, BC.



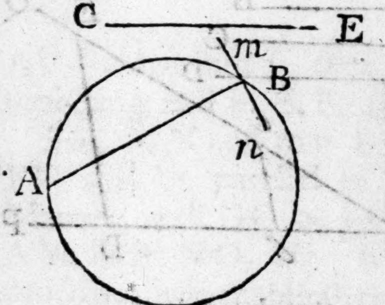
[a] 1. 5.
[b] 6. 5.
[c] Poſt. 3.
[d] 7. 5
[f] 11. 4.

In the infinite Right-line AP take Ab = AB and b C = BC[a]; biſect AC in E[b], and from the Center E, at the Diſtance of EA or (EC) let a Circle ADC be deſcribed[c]; erect bD perpendicular to AC[d] cutting the Circle in D; then will bD be the Mean-proportional required.

For, Ab (AB) : bD : : bD : bC (BC)[f]. *Which was to be done.*

PROBLEM XV.

To inſcribe, in a given Circle, a Right-line AB equal to any given Right-line CE, leſs than the Diameter of the Circle.

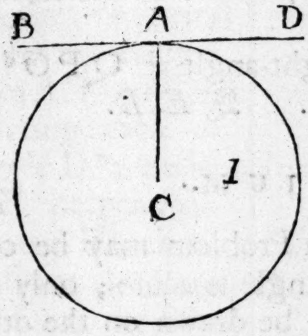


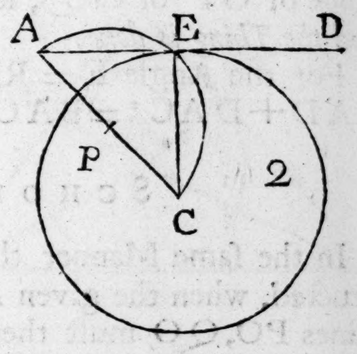
[b] Poſt. 3.
[c] 8. 3.
[d] Def. 29. 1.

From any Point A in the Circumference, with the Radius CE, let a Circle m n be deſcribed[b], cutting the given Circle in B[c]; draw AB, *and the Thing is done*[d].

PROBLEM XVI.

To draw a Tangent to a given Circle C, thro' a given Point A.





a Poſt. 1.
b 7. 5.
c Def. 9. of 3.

Case 1. If the given Point be in the Circumference; then, to the Center C, draw AC[a], and perpendicular to AC[b] draw AB, which will touch the Circle at A[c].

d 6. 5.
e Poſt. 3.
f 8. 3.
g 11. 3.

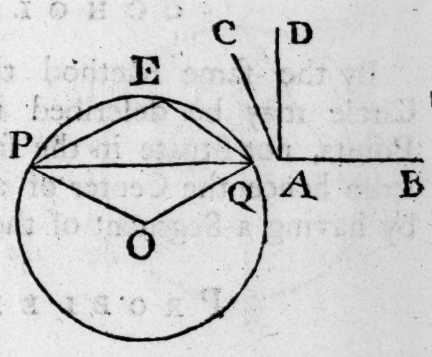
Case 2. If the Point A be without the Circumference; then draw AC[a], which biſect in P[d], and from the Center P, at the Diſtance of AP (or CP), let a Semi-circle AEC be deſcribed[e], cutting the given Circle in E[f]; then draw AE, which will be the Tangent required[c]: Becauſe (CE being drawn) AEC will be a Right-angle[g].

Problem XVII.

Upon a given Right-line PQ to deſcribe the Segment of a Circle PEQ, to contain an Angle E equal to a given Angle BAC.

a 7. 5.
b 4. 5.

Make AD perpendicular to AB[a], alſo make PQO and QPO, each, equal to DAC[b] (the Difference between the given Angle and a right one); then upon the Point of Interſection O, as a Center, at the Diſ-



tance of OP (or OQ), let a Circle be deſcribed[c]; *and the Thing is done.*

For the Angle E = Right-angle + QPO[d] = BAD + DAC[e] = BAC[f]. *Q. E. D.*

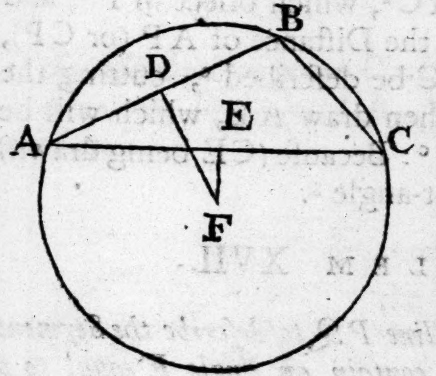
[c] Poſt. 3.
[d] 12. 3.
[e] Ax. 4. of 1.
[f] Ax. 3. of 1.

SCHOLIUM.

In the ſame Manner the Problem may be conſtructed, when the given Angle is acute; only the Lines PO, QO muſt then be drawn on the other Side of PQ, as is manifeſt from Theor. XII. of 3.

PROBLEM XVIII.

About a given Triangle ABC to deſcribe a Circle.



Let any two Sides AB and AC be biſected by two Perpendiculars DF and EF[a]; and they will interſect each other in F, the Center of the required Circle[b].

[a] 6. 5.
[b] 14. 3.

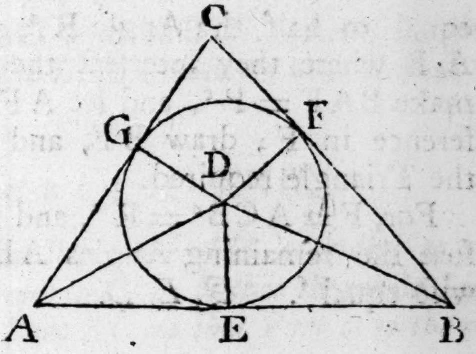
SCHOLIUM.

By the ſame Method the Circumference of a Circle may be deſcribed thro' any three given Points, not ſituate in the ſame Right-line. Alſo, from hence the Center of a Circle may be found, by having a Segment of the Circle given.

PROBLEM XIX.

To inſcribe a Circle in a given Triangle ABC.

Biſect



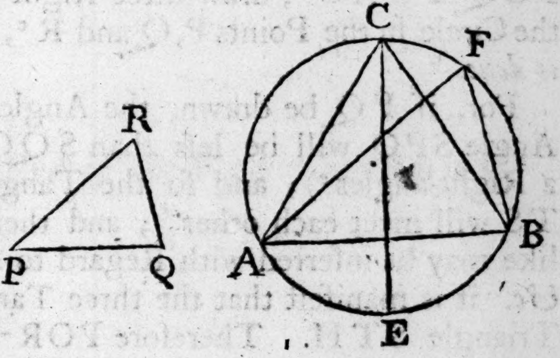
Biſect any two of the Angles A and B, by the Right-lines AD and BD[a], meeting each other in D[b]; make DE perpendicular to AB[c]; then, if from the Center D, at the Diſtance of DE, a Circle be deſcribed, it will touch all the Sides of the Triangle.

For, let DG and DF be perpendicular to AC and BC[c]; then the Triangles ADE, ADG, having two Angles equal, each to each (by Conſtruction) and AD common, will not only be equiangular[d], but alſo have $DE = DG$[e]. By the ſame Argument is $DE = DF$; therefore the Circumference of the Circle alſo paſſes through G and F[f]; but it touches the Sides of the Triangle in thoſe Points[g], becauſe G and F are Right-angles[h]. *Q. E. D.*

[a] 5. 5.
[b] Ax. 8. 1.
[c] 8. 5.
[d] Cor. 2. to 10. 1.
[e] 20. 1.
[f] Def. 29. of 1.
[g] Def. 9. of 3.
[h] Conſtr.

PROBLEM XX.

In a given Circle ACB, to deſcribe a Triangle, equiangular to a given Triangle PQR.



From any Point C in the Circumference, draw the Right-lines CA and CB, making each an Angle with the Diameter CE

equal

equal to half the Angle R[a], and let the Points A, B where they interſect the Circle be joined; make $BAF = P$[c], and let AF meet the Circumference in F; draw BF, and then ABF will be the Triangle required.

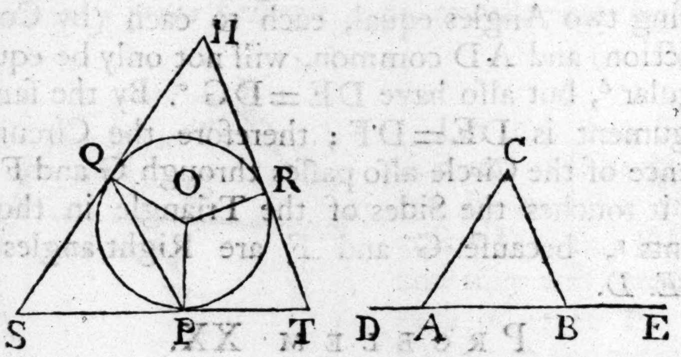
[a] 5. 5. and 4. 5.
[c] 4. 5.
[d] Cor. to 9. 3.

For, $F = ACB$[d] $= R$[e], and $BAF = P$[e]; therefore the remaining Angles ABF and Q are likewiſe equal[f]. *Q. E. D.*

[e] Conſtr.
[f] Cor. 2. to 10, 1.

PROBLEM XXI.

About a given Circle O to deſcribe a Triangle, equiangular to a given Triangle ABC.



Produce the Side AB both ways[a], and at the Center O, make the Angle $POR = EBC$ and $POQ = DAC$[b]; draw three Right-lines to touch the Circle in the Points P, Q and R[c], *and the Thing is done.*

[a] Poſt. 2.
[b] 4. 5.
[c] 16. 5.

For, if PQ be drawn, the Angle $SQP +$ the Angle SPQ will be leſs than $SQO + SPO$ (or 2 Right-angles[e]); and ſo the Tangents HS and TS will meet each other[f]; and therefore, as the like may be inferred with Regard to ST and HT, &c. it is manifeſt that the three Tangents form a Triangle STH. Therefore $POR + T$ being = 2 Right-angles[g] $= ABC + EBC$[h], and $POR = EBC$[k];

[e] Ax. 2. 1. and Def. 9. 3.
[f] Cor. 2. to 8. 1. and Ax. 8. 1.
[g] Cor. 2. to 11. 1.
[h] 4. 1.
[k] Conſtr.

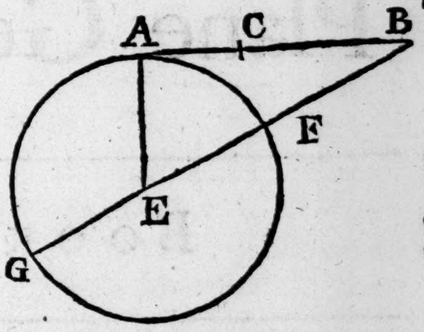
EBC [k]; thence will T = ABC [l]. And, by the ſame Argument S = BAC; whence alſo H = C [l]. *Q. E. D.*

[l] Ax. 5. 1.

PROBLEM XXII.

To divide a given Right-line AB, according to extreme and mean Proportion; or ſo that the whole Line AB may be to one Part BC, as that Part is to the remaining Part AC.

[a] 7. 5.
[b] 6. 5.



Make AE perpendicular to AB [a], and equal to $\frac{1}{2}$ AB [b], and from the Center E, at the Diſtance of EA, let a Circle be deſcribed, and draw BEG, cutting the Circumference in F and G; then make BC = BF, *and the Thing is done.*

[c] Cor. to 17. 3.
[d] 3. 4.
[e] 8. 4.

For, ſince $AB^2 = BG \times BF$ [c], therefore AB : BG :: BF : AB [d]; whence dividedly, AB : BF (BC) :: BF (BC) : AC [e]. *Q. E. D.*

The End of the FIFTH BOOK.

ELE-

ELEMENTS

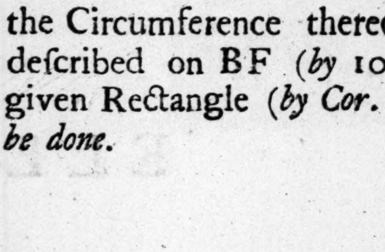
OF

Plane Geometry

BOOK VI.

PROBLEM I.

To make a Square equal to a given Rectangle ABCD.

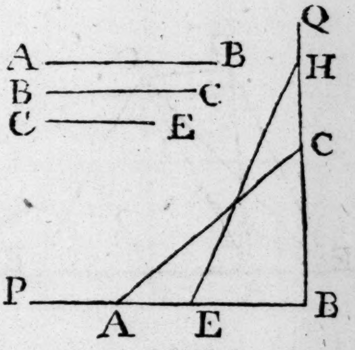


In AB produced take $BE = BC$[a], biſect AE in O[b], and from the Center O, at the Diſtance of OA, or OE, let a Circle AFE be deſcribed, and let CB produced meet the Circumference thereof in F[c]; then a Square deſcribed on BF (*by* 10. 5.) will be equal to the given Rectangle (*by Cor. to* 11. 4.) *Which was to be done.*

a 1. 5.
b 6. 5.
c Cor. to 3. 3.

PROBLEM II.

To make a Square equal to any Number of given Squares.

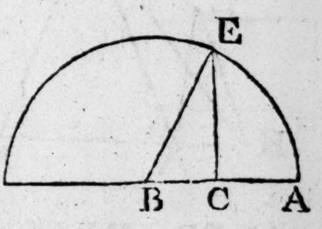


Let AB, BC, CE be the Sides of the given Squares. Draw two infinite Lines BP, BQ at Right-angles to each other[a]; in which take BA = BA, BC = BC and join A, C: Then will $AC^2 = AB^2 + BC^2$ [b]. Again, in the ſame Right-lines take BH = AC, BE = CE and join E, H: Then will $EH^2 = BH^2 + BE^2$ [b] $= AC^2 + CE^2$ [c] $= AB^2 + BC^2 + CE^2$ [d]. Therefore a Square made upon EH (by 10. 5.) will be equal to the three given Squares. *Which was to be done.*

[a] 7. 5.
[b] 7. 2.
[c] Hyp. and Cor. to 1. 2.
[d] Ax. 3. 1.

PROBLEM III.

To make a Square equal to the Difference of two given Squares.

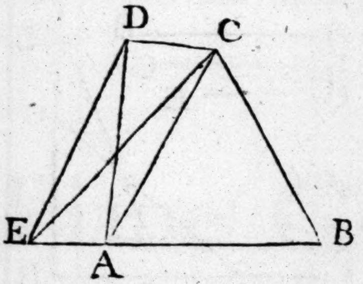


Let AB and BC be the Sides of the two given Squares. Upon the Center B, with the Radius AB, let a Circle be deſcribed, and make CE perpendicular to BC[a], meeting the Circumference thereof in E[b]. Then is $CE^2 = BE^2$ $(AB^2) - BC^2$ [c]. *Which was to be done.*

[a] 7. 5.
[b] Cor. to 3. 3.
[c] Cor. to 7. 2.

PROBLEM IV.

To make a Triangle BCE equal to a given Quadrilateral ABCD.



[a] 9. 5.

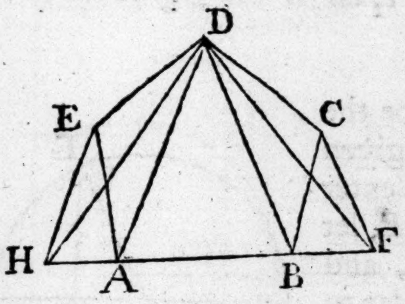
Draw the Diagonal AC, alſo draw DE parallel to AC[a], meeting BA produced in E; then draw CE, *and the Thing is done.*

[b] Cor. 1. to 2. 2.
[c] Ax. 4. 1.

For the Triangles ACD, ACE, being upon the ſame Baſe AC, and between the ſame Parallels AC and ED, are equal[b]; therefore, if ABC be added to each, then alſo will ABCD = BCE[c]. *Q. E. D.*

PROBLEM V.

To make a Triangle DHF equal to a given five-ſided Figure ABCDE.



[a] 9. 5.

Draw DA and DB, and alſo EH and CF parallel to them[a], meeting AB produced in H and F; then draw DH and DF, *and the Thing is done.*

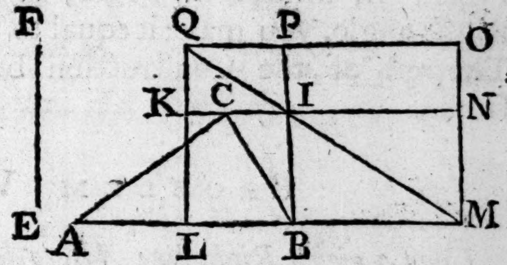
[b] Cor. 1. to 2. 2.
[c] Ax. 4. and 1. 1.

For the Triangle DEA = DHA[b], and DCB = DFB[b]; therefore ABCDE (= DEA + DCB + ABD = DHA + DFB + ABD) = DHF[c]. *Q. E. D.*

PROBLEM

PROBLEM VI.

Upon a given Right-line EF, to make a Rectangle equal to a given Triangle ABC.



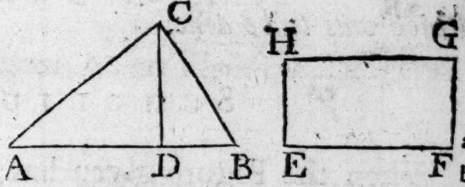
Thro' C draw KN parallel to AB[a], and biſect AB with the Perpendicular LQ[b] meeting KN in K; alſo, draw BP, perpendicular to AB[b], interſecting KN in I; then in AB, produced, take BM = EF, and draw MIQ cutting LQ in Q, alſo draw QO and MO, parallel to AM and LQ[a], meeting each other in O: Then will INOP be the Rectangle required.

[a] 9. 5.
[b] 7. 5.

For, it is evident, that LI, IO and LO are all Rectangles[d]: Therefore IN = BM[e] = EF[f], and IO = LI[g] = ABC[h]. *Q. E. D.*

[d] Def 23. and Cor. to 23. 1.
[e] 23. 1.
[f] Conſtr.
[g] 3. 2.
[h] Cor. 2. to 2. 2.

The ſame otherwiſe.



From any Angle C, upon the oppoſite Side AB, let fall the Perpendicular CD[a]; make EH, perpendicular to EF[b], and equal to a Fourth-proportional to 2 EF, AB, and CD[c]: Then the Rectangle EG contained under EF and EH will be equal to the Triangle ABC.

For, ſince, by Conſtruction, 2 EF : AB :: CD : EH, therefore is 2 EF × EH = AB × CD[d], and conſequently EF × EH = $\frac{1}{2}$ AB × CD = ABC[f]. *Q. E. D.*

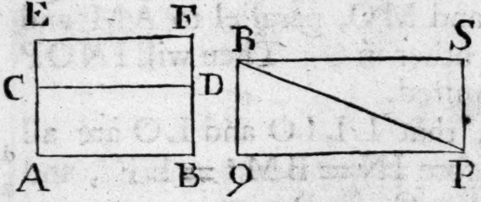
[a] 8. 5.
[b] 7. 5.
[c] 13. 5.
[d] 3. 4.
[f] Cor. 2. to 2. 2.

SCHOLIUM.

By the firſt of the two preceding Methods, a Parallelogram, having a given Angle, may be deſcribed, upon a given Line, equal to a given Triangle; if, inſtead of MBP, or MLQ, being a Right-angle, you make it equal to the Angle given: The reſt of the Conſtruction being exactly the ſame.

PROBLEM VII.

Upon a given Right-line AB, to deſcribe a Rectangle equal to a given right-lined Figure PQRS.



Let the given Figure be divided into Triangles PQR, PRS: And upon AB (by the precedent) let a Rectangle ABDC, equal to the Triangle PQR, be deſcribed; alſo, upon CD, make the Rectangle CDFE equal to the Triangle PRS: Then, alſo, will ABFE (= PQR + PRS[a]) = PQRS[b]. *Which was to be done.*

[a] Ax. 3. and 4. 1.
[b] Ax. 3. 1.

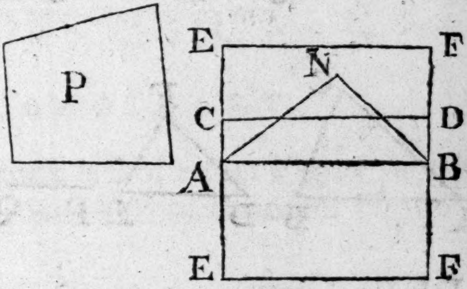
SCHOLIUM.

When the Figure given has not more than five Sides, the Conſtruction will be more eaſy, by firſt finding a Triangle equal to it (as in Prob. 4. and 5.) and then making a Rectangle equal to that Triangle. But if the propoſed Figure be a Rectangle, the eaſieſt Way, of all, will be to take a Fourth-proportional BF to the given Line AB and the two Sides PQ and PS of the given Rectangle (by

13. 5); which Fourth-proportional will be the Altitude of the Rectangle required. For, ſince $AB : PQ :: PS : BF$ (*by Conſtr.*) therefore (by 3. 4.) $AB \times BF = PQ \times PS$.

PROBLEM VIII.

To find the Sum, Difference, or Ratio, of any two given right-lined Figures ABN and P.



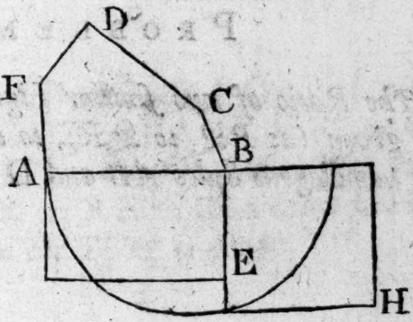
By the precedent, let two Rectangles AD and AF, reſpectively equal to ABN and P, be deſcribed, on the ſame or different Sides of AB, according as the Difference, or Sum, of the two Figures is required: Then will the Rectangle CF expreſs that Sum or Difference [b]; and $AE : AC :: AF\ (P) : AD\ (ABN$[c].) *Which was to be done.*

[b] Ax. 4. or 5. 1.

[c] 1. 4.

PROBLEM IX.

To make a Square equal to any right-lined Figure given ABCDF.



Upon AB deſcribe a Rectangle AE equal to ABCDF [a]; then make a Square BH equal to that Rectangle [b], *and the Thing is done.*

[a] 7. 6.

[b] 1. 6.

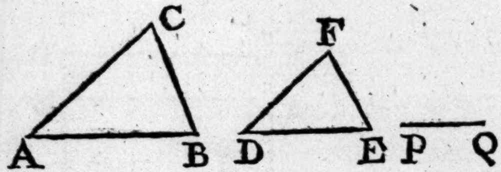
SCHO-

SCHOLIUM.

After the ſame Manner (*from Prob.* 8.) a Square may be deſcribed equal to the Sum, or Difference, of any two given right-lined Figures.

PROBLEM X.

To find two Lines having the ſame Ratio to each other, as two given ſimilar Figures ABC, DEF.



a 12. 5. Find a Third-proportional PQ to any two homologous Sides AB, DE[a]: Then will AB:PQ::ABC : DEF.

For, by Conſtruction, AB : DE :: DE : PQ,
b 3. 4. therefore AB × PQ = DE^2 [b]; but AB : PQ (::
c 1. 4. AB^2 : AB × PQ [c] :: AB^2 : DE^2 [d]) :: ABC : DEF[e].
d Ax. 2. 4. *Q. E. D.*
e 19. 4.

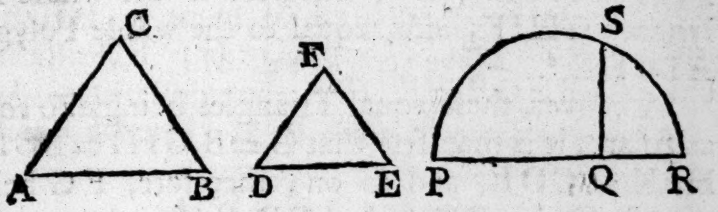
COROLLARY.

Hence, it appears, that ſimilar right-lined Figures are in the duplicate Ratio of their homologous Sides.

PROBLEM XI.

The Ratio of two ſimilar Figures ABC, DEF being given (as PQ to QR), to determine the Ratio of the homologous Sides AB and DE.

Find



Find a Mean-proportional QS between PQ and QR[a]: Then AB : DE :: PQ : QS. For $AB^2 : DE^2$ (:: ABC : DEF[b] :: PQ : QR[c]) :: $PQ^2 : PQ \times QR$[d]; but $PQ \times QR = QS^2$[e]; therefore $AB^2 : DE^2 :: PQ^2 : QS^2$[f], and conſequently AB : DE :: PQ : QS[g]. *Q. E. D.*

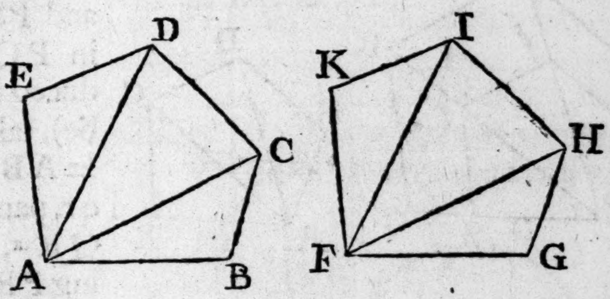
[a] 14. 5.
[b] 19. 4.
[c] Hyp.
[d] 1. 4.
[e] 3. 4.
[f] Ax. 2. of 4.
[g] Cor. 2. to 9. 4.

COROLLARY.

Hence, if one Side AB of either Figure be given the correſponding Side DE of the other Figure will be known.

PROBLEM XII.

To deſcribe a Figure FGHIK equal, and ſimilar, to a given right-lined Figure ABCDE.



Draw AC and AD, and alſo FG equal to AB[a]; make the Angle GFH = BAC[b], HFI = CAD and IFK = DAE[b]; likewiſe make FH = AC, FI = AD, and FK = AE[a]; then draw GH, HI, IK and KF, *and the Thing is done.*

[a] 2. 5.
[b] 4. 5.
[c] Ax. 9. of 1.
[d] Ax. 4. 1.
[f] Def. 8. 4.

For, ſince the Triangle FGH = ABC [c] and FHI = ACD [c], &c; therefore is the whole Polygon FGHIK, alſo, equal to the whole Polygon ABCDE [d].

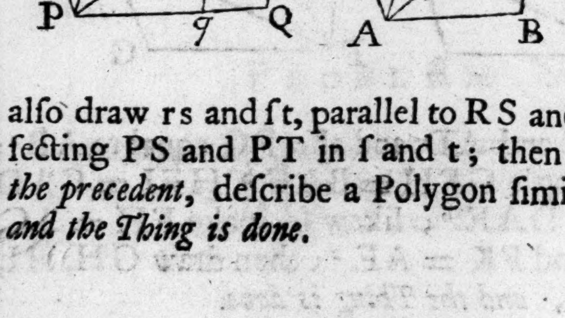
Moreover, theſe equal Triangles being alſo equiangular [e], it is manifeſt, that G = B, GHI = BCD [d], HIK = CDE, and ſo on; therefore, FG being = AB, GH = BC, HI = CD [c], &c. the two Polygons ABCDE, FGHIK are like to each other [f]. *Q. E. D.*

SCHOLIUM.

The Figure FGHIK may be otherwiſe conſtructed, by making the Triangles FGH = FHI, &c. reſpectively; as is evident from 19. 1. and Ax. 4.

PROBLEM XIII.

Upon a given Right-line AB, to deſcribe a Figure ABCDE ſimilar to a given right-lined Figure PQRST.



Draw PR and PS, and in PQ (produced if need be) take Pq = AB; draw qr, parallel to QR [a], meeting PR in r; alſo draw rs and ſt, parallel to RS and ST [a], interſecting PS and PT in ſ and t; then upon AB, *by the precedent*, deſcribe a Polygon ſimilar to Pqrſt, *and the Thing is done.*

[a] 9. 5.
[b] Cor. 1. to 8. 1. and Ax. 1. 1.
[c] 13. 4.
[d] Def. 8. 4.
[e] Ax. 2. 1.

For,

For, ſince any Angle BCD (q r s) of the Polygon ABCDE is equal to its correſpondent QRS[b]; and alſo CB (q r) : CD (r s) :: RQ : RS[c], therefore the two Polygons ABCDE, PQRST are like to each other[d]. *Q. E. D.*

COROLLARY.

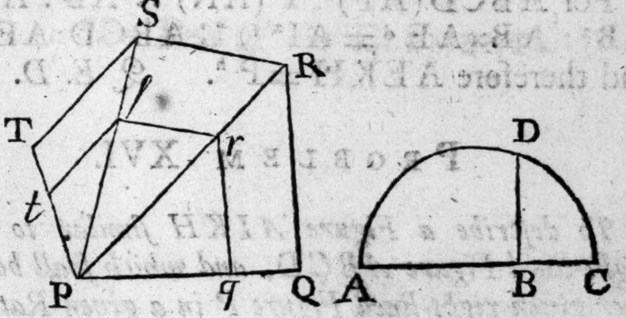
Hence, it follows, that of two right-lined Figures PQRST and ABCDE (Pqrſt) like, and alike ſituate, that is the greateſt whoſe Baſe is the greateſt, and *vice verſa*[e].

SCHOLIUM.

This laſt Problem may likewiſe be conſtructed by making the Triangles ABC, ACD, ADE equiangular to the Triangles PQR, PRS, PST reſpectively.

PROBLEM XIV.

To make a Figure Pqrſt, ſimilar to a right-lined Figure given PQRST, which ſhall alſo be to it in a given Ratio; viz. *as BC to AB.*

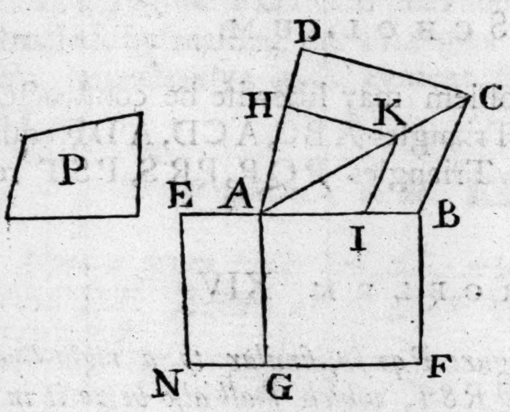


The Ratio of the two Figures, and the Baſe PQ of one of them being given, the Baſe Pq, of the other,

other, will alſo be given, *by* 11. 6. (that is, by taking BD a Mean-proportional between AB and BC, and Pq a Fourth-proportional to AB, BD and PQ.) Therefore, if, upon Pq thus determined, a Polygon Pqrſt be deſcribed *(by the precedent)* ſimilar to PQRST, *the Thing is done.*

PROBLEM XV.

To deſcribe a Figure AIKH equal to one right-lined Figure given P, and ſimilar to another ABCD.



Upon AB make the Rectangle ABFG equal to ABCD[a], and upon AG make the Rectangle AGNE = P[a]; in AB take AI equal to a Mean-proportional between AB and AE[b]; and upon AI let AIKH be deſcribed ſimilar to ABCD[c], *and the Thing is done.*

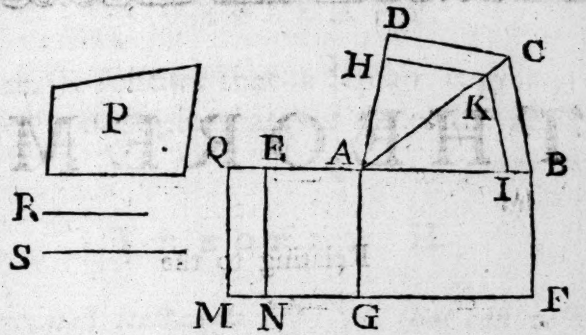
For ABCD(AF) : P(AN) :: AB : AE[d] (:: AB^2 : AB×AE[d] = AI^2 [f]) :: ABCD:AEKH[g]; and therefore AEKH = P[h]. *Q. E. D.*

[a] 7. 6.
[b] 14. 5.
[c] 13. 6.
[d] 1. 4.
[f] Conſtr. and 3. 4.
[g] 19. 4.
[h] Ax. 1. 4.

PROBLEM XVI.

To deſcribe a Figure AIKH ſimilar to a given right-lined Figure ABCD, and which ſhall be to another given right-lined Figure P in a given Ratio; viz. *as S to R.*

Make



Make the Rectangle ABFG = ABCD [a], and the Rectangle AGNE = P [a]; alſo in AE, produced, take AQ = a Fourth-proportional to R, S and AE [b]; then (*by the precedent*) make AIKH equal to the Rectangle AG × AQ, and ſimilar to ABCD, *and the Thing is done.*

For, R : S :: AE : AQ [c] :: [d] AN (P) : AM (AIKH.) *Q. E. D.*

[a] 7. 6.
[b] 13. 5.
[c] Conſtr.
[d] 1. 4.

The End of the ELEMENTS.

THEOREMS

Relating to the

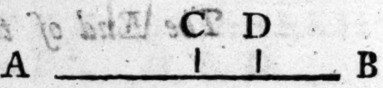
MAXIMA and MINIMA

OF

LINES, ANGLES and SURFACES.

THEOREM I.

Of all the Rectangles AC × BC, AD × BD that can poſſibly be contained under the two Parts of a given Right-line AB, any wiſe divided, that AC × BC will be the greateſt whoſe Sides AC, BC are equal to each other.



FOR, ſince AD (AC + CD) × BD (AC — CD) is $= AC^2 - CD^2$ [a], it is manifeſt that AC^2 (AC × BC [b]) is $= AD \times BD + CD^2$, or that AC × BC is greater than AD × BD by CD^2. *Q. E. D.*

[a] 6. 2.

[b] 1. 2. and Ax. 4. 1.

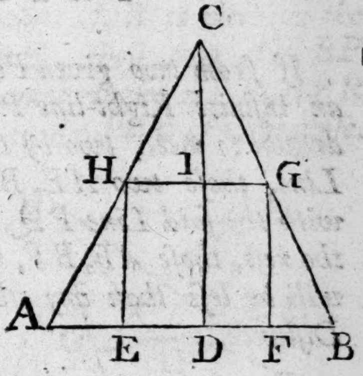
COROL-

COROLLARY.

Hence, it follows that a Square is greater than any other Rectangle whatever under the ſame Perimeter.

THEOREM II.

The greateſt Rectangle EFGH, that can poſſibly be inſcribed in a given Triangle ABC, is that whoſe Altitude DI is equal to half the Altitude CD of the Triangle.



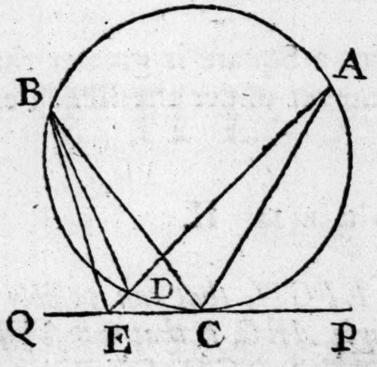
For AB : CD :: HG : CI [a] :: HG × DI : CI × DI [b]; therefore, HG × DI, or the Rectangle FH, being in a conſtant Ratio to CI × DI, namely as AB to CD, it is evident that the ſaid Rectangle FH will be the greateſt poſſible when CI × DI is the greateſt poſſible; that is, when CI and DI are equal to each other [c].

[a] 12. 4.
[b] 1. 4.
[c] Theor. 1.

THEOREM III.

If from any two Points A and B, in the Periphery of a given Circle ABC, ſeveral Right-lines AC, BC; AE, BE be drawn to meet, two and two, upon a Right-line PQ touching the Circle, thoſe two AC and BC will contain the greateſt Angle, which meet in C the Point of Contact.

For,



[a] 5. 3.

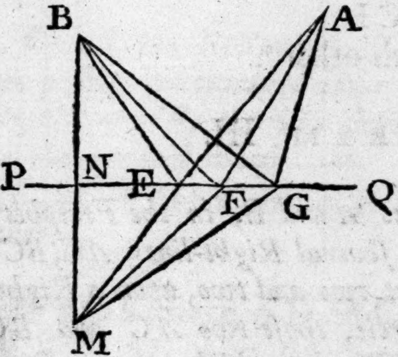
[b] 10. 1.

[c] Cor. to 9. 3.

For, ſince the Point E falls without the Circle[a], AE will cut the Circumference in ſome Point D; and therefore, if DB be drawn, the Angle BED will be leſs than BDA[b], or than its Equal ACB[c]. *Q. E. D.*

THEOREM IV.

If from two given Points A, B, on the ſame Side of an infinite Right-line PQ, any Number of Lines be drawn to meet, two by two, in different Points of that Line, thoſe two AE, BE, which make equal Angles with the ſaid Line PQ, will be the leaſt of all; and, of the reſt, thoſe AF, BF, which are neareſt the two leaſt, will be leſs than any other two AG, BG at a greater Diſtance.



[a] 5. 1.
[b] Hyp.
[c] Conſtr.
[d] 20. 1.
[e] Ax. 9. 1.
[f] 15. 1.
[g] 17. 1.

For, let BNM be perpendicular to PNQ, and let AE be produced to meet it in M, alſo let MF and MG be drawn:

Since MEN = AEQ[a] = BEN[b], MNE = BNE[c], and NE common, therefore is MN = BN, and ME = BE[d]; whence alſo MF = BF, and MG = BG[e]: But AM (AE + BE) is leſs than AF + MF, or its Equal AF + BF[f], and AF + BF (AF + MF) leſs than AG + MG, or its Equal AG + BG[g]. *Q. E. D.*

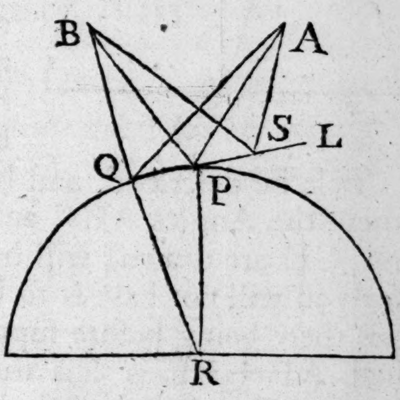
COROLLARY.

Hence, it appears that the Concourſe E of the two leaſt, with reſpect to the Concourſe F of any other two, will be on that Side where the Angle is leaſt: For BFP[a] $\sqsupset$ BEP (or AEQ[b]) $\sqsupset$ AFQ[a].

[a] 10. 1.
[b] Hyp. and Ax. 6. 1.

THEOREM V.

Of all the Right-lines AP, BP; AQ, BQ that can be drawn from two given Points A, B to meet on the Convexity of a given Circle PQ, thoſe two AP, BP will be the leaſt which make equal Angles APR, BPR with the Radius RP drawn to the Point of Concourſe.



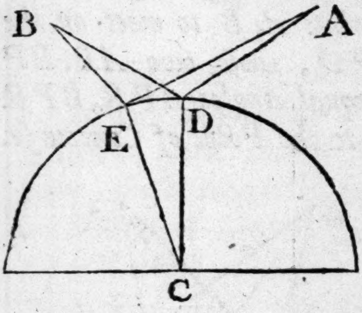
Draw the Radius RQ, and thro' Q and P draw the infinite Right-line QPL, and let S be that Point therein where the Lines AS and BS muſt meet to be the leaſt poſſible, or where the Angles ASL and BSQ are equal to each other[a].

The Angle LPR is $\sqsubset$ RQP or RPQ[b], and therefore, APR being = BPR[c], LPA is $\sqsupset$ QPB[d], and ſo, the Point P falling on the ſame Side of S with Q *(by Corol. to the precedent)*, SP will be leſs than SQ[e], and conſequently AP + BP leſs than AQ + BQ[f]. *Q. E. D.*

[a] Theor. 4.
[b] 6. and 10. 1.
[c] Hyp.
[d] Ax. 6. 1.
[e] Ax. 2. 1.
[f] Theor. 4.

THEOREM VI.

If from three given Points A, B, C it be required to draw as many Right-lines AD, BD, CD to another Point D (ſituate ſomewhere between them), ſo that the Sum of all theſe three together may be the leaſt poſſible; then muſt the Poſition of that Point be ſuch that all the Angles about it, ADB, BDC, CDA, may be equal to each other.



For, if you ſay the Sum of AD, BD and CD is the leaſt poſſible, and yet the Angles not equal: Then let ADC and BDC be unequal, and upon the Center C, at the Diſtance of CD, let the Circumference of a Circle be deſcribed, and let E be that Point in it, where the Angles AEC and BEC (when AE, BE and CE are drawn) will be equal to each other, (for you will not have it to be the Point D, the Angles there being before ſuppoſed unequal). Then, ſince $AE + BE$ is leſs than $AD + BD$[b], therefore is $AE + BE + CE$ alſo leſs than $AD + BD + CD$[c]; which is abſurd: Therefore all the Angles are equal. *Q. E. D.*

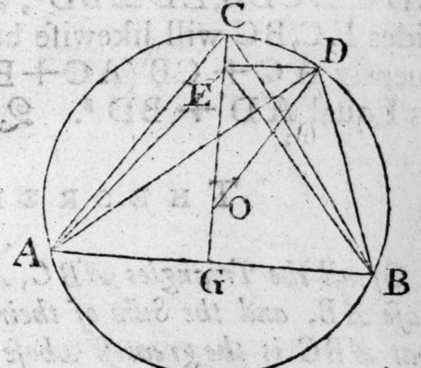
[b] Theor. 5.

[c] Ax. 6. 1.

THEOREM VII.

Of all the Triangles ABC, ABD, in the ſame Segment of a Circle ACDB, that ABC whoſe Sides AC, BC are equal is the greateſt.

Let CEG be perpendicular, and DE parallel, to AB, and from the Center O, let OD be drawn, and join A, E and B, E.



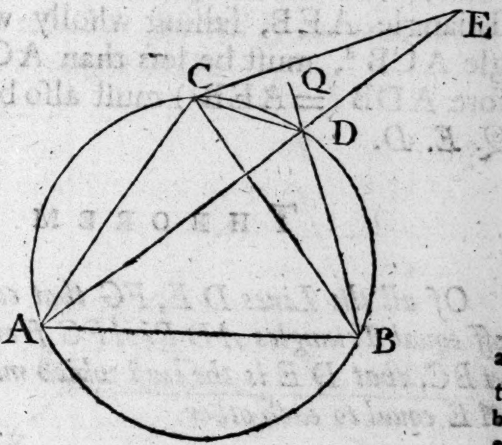
It is evident that CG not only biſects AB[a], but alſo paſſes thro' the Center O[b]: Therefore, OC (OD) being greater than OE[c], the Triangle ACB will alſo be greater than the Triangle AEB or its Equal ADB[d]. *Q. E. D.*

a Cor. 1. to 6. 1.
b Cor. to 14. 3.
c 16. 1.
d Cor. 1. to 2. 2.

THEOREM VIII.

Of all the Triangles ABC, ABD, in the ſame Segment of a Circle ACDB, that ABC has the Sum of its Sides AC + BC the greateſt, which has them equal to each other.

In AD, produced take DE = BD, and let CD and CE be drawn; alſo produce BD to Q.



The Angle QDC is = BAC[a] = ABC[b] = ADC[c]; therefore, if to the firſt and laſt of theſe, be added QDE = ADB[d], the Sums will alſo be equal, that is CDE = CDB; there-

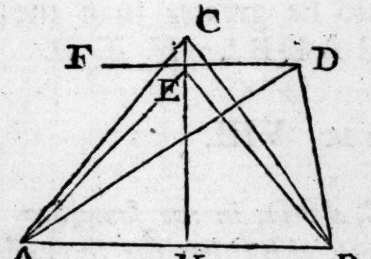
a Cor. 1. to 13. 3.
b 6. 1.
c Cor. to 9. 3.
d 5. 1.

therefore, ſince the Triangles EDC, BDC have $EDC = CDB$, $ED = BD$[e], and CD common, the Sides EC, BC will likewiſe be equal[f], and conſequently $AC + CB\ (AC + EC) \sqsubset AE$, or than its Equal $AD + BD$[g]. *Q. E. D.*

[e] Conſtr.
[f] Ax. 9. 1.
[g] 15. 1.

THEOREM IX.

Of all the Triangles ABC, ABD having the ſame Baſe AB, and the Sum of their other Sides the ſame, that ABC is the greateſt whoſe Sides AC and BC are equal to each other.



Let CH be perpendicular to AB, and DEF, parallel to AB, interſecting HC, produced if need be, in E; likewiſe let AE and BE be drawn.

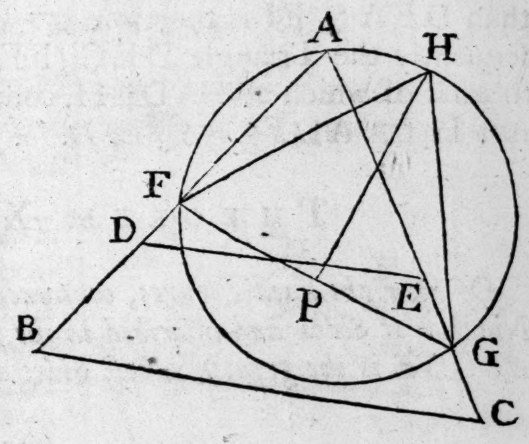
It is manifeſt that the Angles AEF, BED are equal[a]; therefore $AE + BE$ is leſs than $AD + BD$[b], or than its Equal $AC + BC$[c]; and ſo, the Triangle AEB, falling wholly within the Triangle ACB[d], muſt be leſs than ACB[e]; and therefore ADB ($= AEB$[f]) muſt alſo be leſs than ACB. *Q. E. D.*

[a] 7. and 8. 1.
[b] Theor. 4.
[c] Hyp.
[d] 17. 1.
[e] Ax. 2. 1.
[f] Cor. 1. to 2. 2.

THEOREM X.

Of all the Lines DE, FG that can be drawn to cut off equal Triangles ADE, AFG from a given Triangle ABC, that DE is the leaſt which makes the Sides AD, AE equal to each other.

Let A F G be the Circumference of a Circle paſſing thro' A, F, G[p]; alſo let P H be perpendicular to the Middle of F G meeting the Circumference in H, and join F, H and G, H.



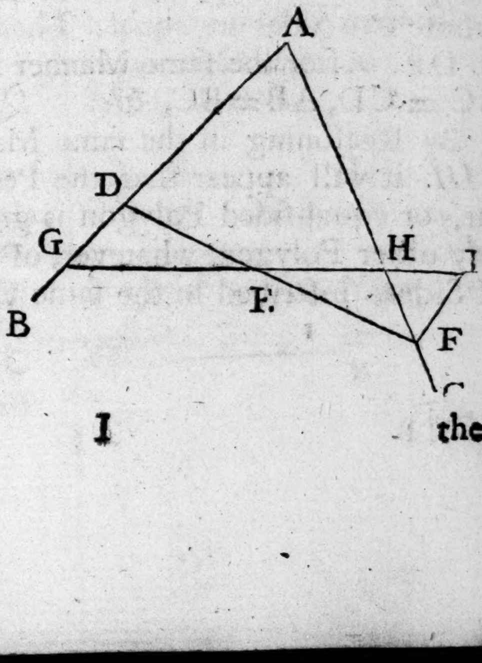
It is evident that F H is = G H[a], and, conſequently, that the Triangle G H F is greater than A F G or its Equal A D E[b]; therefore, as the Triangles G H F, A D E are equiangular[c], the Baſe of the greater F G will alſo be greater than D E the Baſe of the leſſer[d]. *Q. E. D.*

[p] 14. 3.
[a] Ax. 9. 1.
[b] Theor. 9.
[c] Cor. to 9. 3. and Cor. 8. to 10. 1.
[d] Cor. to 13. 6.

THEOREM XI.

Of all the Right-lines D E F, G E H that can be drawn thro' a given Point E, between two Right-lines A B, A C; that D F, which is biſected by the ſaid Point E, cuts off the leaſt Triangle A D F.

Let F I be parallel to A B interſecting G H, produced if need be, in I.



The Triangles F E I, D E G, having F E = D E[a], I F E = G D E[b], F I E = D G E[b], are equal[c]; but

[a] Hyp.
[b] 8. 1.
[c] Cor. to 30. 1.

the Angle DFI (or its Equal BDF) being greater than DFA [d], EI is therefore ⊏ EH [e], and conſequently the Triangle DEG (FEI) ⊏ EFH [e]; to each of which add ADEH, common, and then AGH ⊏ ADF [f]. *Q. E. D.*

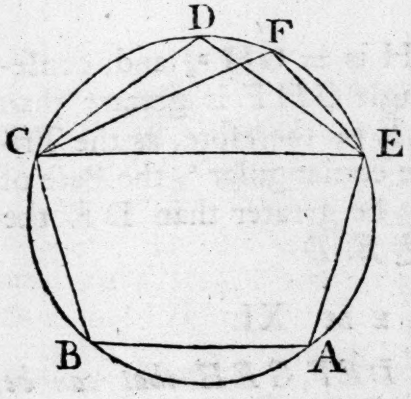
[d] Cor. 6. to 10. 1.

[e] Ax. 2. 1.

[f] Ax. 6. 1.

THEOREM XII.

Of all right-lined Figures, contained under the ſame Number of Sides and inſcribed in the ſame Circle, that ABCDE is the greateſt whoſe Sides are all equal.

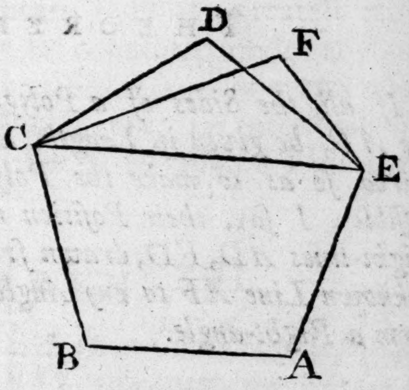


For, the Polygon ABCDE being the greateſt poſſible, the Triangle CDE muſt be greater than any other Triangle CFE in the ſame Segment; for if CFE were equal to, or greater than CDE, then the whole Polygon ABCFE would be either equal to, or greater than the whole Polygon ABCDE, which is contrary to Hypotheſis. But, *by Theor. VII.* the greateſt Triangle, in the ſame Segment, is that whoſe two Sides are equal: Therefore CD is equal to DE. After the ſame Manner it will appear that BC = CD, AB = BC, *&c.* *Q. E. D.*

By Reaſoning in the ſame Manner, *from Theor. VIII.* it will appear that the Perimeter of a regular, or equal-ſided Polygon is greater than that of any other Polygon, whatever, of the ſame Number of Sides, inſcribed in the ſame Circle.

THEOREM XIII.

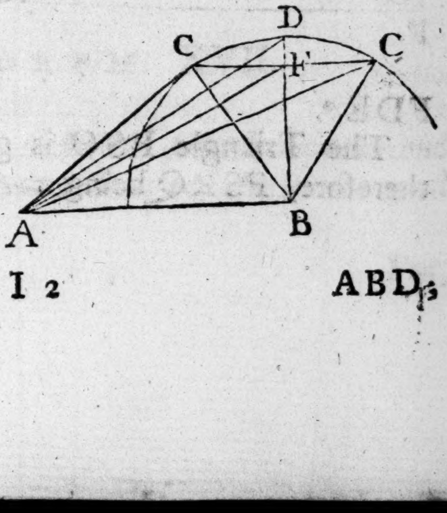
Of all right-lined Figures, contained under the ſame Perimeter and Number of Sides, the greateſt is when the Sides are all equal.



For, if ABCDE be the greateſt poſſible, the Triangle CDE muſt, manifeſtly, be greater than any other Triangle CFE, upon the ſame Baſe, whereof the Sum of the other Sides is alſo the ſame. But, by *Theorem IX.* the greateſt Triangle, when the Baſe and the Sum of the Sides are given, is that whoſe Sides are equal: Therefore ED and DC are equal. In the ſame Manner it will appear that BC = CD, *&c.* *Q. E. D.*

THEOREM XIV.

The greateſt Triangle that can poſſibly be contained under two Right-lines AB, BD, given in Length, and any other Right-line AD, joining their Extremes, is when the two given Lines make right Angles with each other.



Let BC be any other Poſition of BD; or, which is the ſame, let BC be equal to BD, and the Angle ABC greater or leſs than

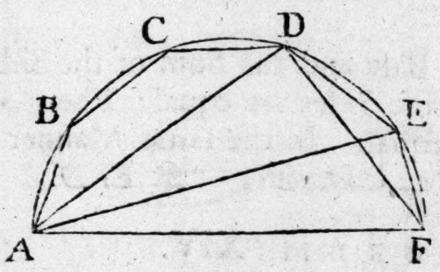
ABD; alſo let CF be parallel to AB meeting BD produced, if need be, in F, and draw AF.

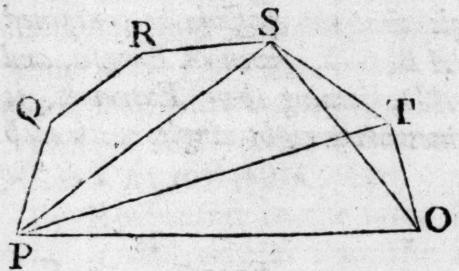
It is evident that BFC is a Right-angle [a], and that BD (BC) is greater than BF [b]; therefore, the Triangle ADB, being greater than ABF [c], is alſo greater than its Equal ABC [d]. *Q. E. D.*

[a] 1. 1.
[b] 16. 1.
[c] Ax. 2. 1.
[d] Cor. 1. to 2. 2. and Ax. 5. 1.

THEOREM XV.

If all the Sides of a Polygon ABCDEF, except one AF, be given in Length, and their Poſition be required ſo as to make the Polygon it ſelf the greateſt poſſible; I ſay, their Poſition muſt be ſuch that two Right-lines AD, FD, drawn from the Extremes of the unknown Line AF to any Angle D of the Polygon, may form a Right-angle.





For, if you would have ABCDEF to be the greateſt poſſible, and yet ADF not a Right-angle: Then let PSO be a Right-angle, contained under PS = AD, and OS = FD; and, upon PS and OS, let the Figures PSRQ and OST be deſcribed equilateral, and equiangular, to ADCB and FDE [a].

The Triangle PSO is greater than ADF [b]; therefore, PSRQ being = ADCB, and OST = FDE [a],

[a] 12. 6.
[b] Theor. 14.

FDE [a], the whole Polygon PQRSTO is alſo greater than the whole Polygon ABCDEF [c]. *Which is abſurd.* [c] Ax. 6. 1.

COROLLARY.

Hence, becauſe the Angle in a Semi-circle is a Right-angle [d], it appears that the greateſt right-lined Figure, that can be contained under any propoſed Number of given Sides and one other Side any how taken, is when it may be inſcribed in a Semi-circle, whereof the indetermined Line is the Diameter. [d] 11. 3.

THEOREM XVI.

A Polygon ABCDEA in a Circle is greater than any other Polygon PQRST, whatever, whoſe Sides are the ſame both in Length and Number. See the preceding Figure.

Let AF be the Diameter of the Circle, and join E, F; alſo make the Angle PTO = AEF, TO = EF, and let PO be drawn.

Firſt, (not regarding the Baſes AF, PO) it is plain, becauſe the ſeveral Sides of the two Figures are reſpectively equal to each other, that the whole Polygon ABCDEF is greater than the whole Polygon PQRSTO [b]; therefore, if from theſe the equal Triangles AEF, PTO [c] be taken away, there will remain ABCDEA $\sqsubset$ PQRSTP [d]. *Q. E. D.* [b] Theor. 15. [c] Ax. 9. 1. [d] Ax. 6. 1.

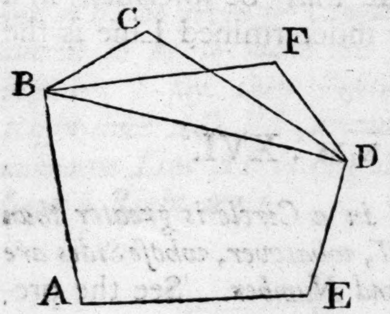
THEOREM XVII.

Of all the Polygons, under the ſame Perimeter and Number of Sides, that whoſe Sides and Angles are equal is the greateſt.

For, the greateſt Polygon that can be contained under a given Perimeter, is one whoſe Sides are all equal[a]. But of all the Polygons of this Sort, which differ only in their Angles, that is the greateſt which may be inſcribed in a Circle[b]: Therefore the greateſt, of all, is that whoſe Sides are all equal and which may be inſcribed in a Circle, or whereof the Angles, as well as the Sides, are all equal. *Q. E. D.*

[a] Theor. 13.

[b] Theor. 14.

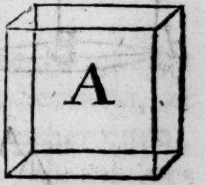


Note, The Magnitude of the greateſt Polygon, that can be contained under any Number of unequal Sides, does not at all depend upon the Order in which thoſe Lines are placed with regard to each other. For let ABCDE be the greateſt, one way, or according to one Order of the Sides; and upon BD let a Triangle BDF be conſtituted whoſe Sides DF and BF are, reſpectively, equal to BC and DC; then, the Triangles BCD, BFD being equal, the whole Polygons ABCDE and ABFDE will likewiſe be equal, notwithſtanding their equal Sides BC, DF, &c. are placed according to different Orders.

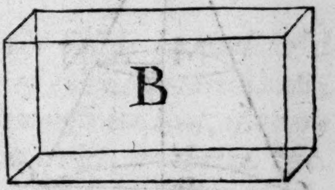
A
TREATISE
OF
REGULAR SOLIDS.

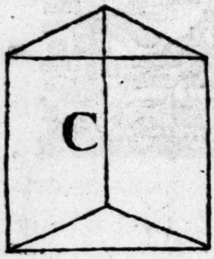
DEFINITIONS.

1. A CUBE is a Solid bounded by ſix equal Squares ſtanding at Right-angles; as A.

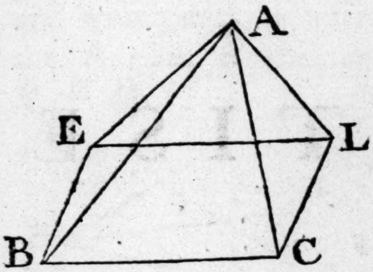


2. A Parallelopipedon is a Solid bounded by ſix Rectangles ſtanding at Right-angles; as B.

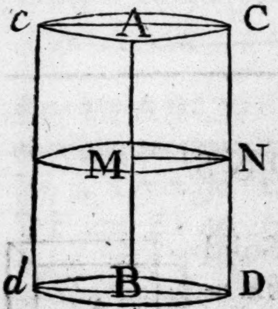




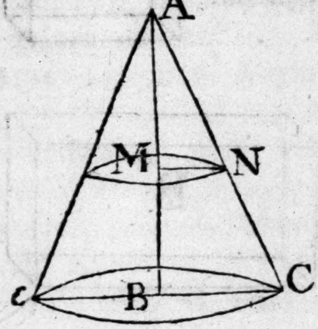
3. A Priſm is a Solid whoſe Sides are Parallelograms, and whoſe two Ends are parallel to each other; as C.



4. A Pyramid is a Solid whoſe Baſe is a Polygon or right-lined Figure, and whoſe Sides are Triangles meeting in a Point, above the Baſe, call'd the Vertex of the Pyramid: Thus ABCLE repreſents a Pyramid whoſe Vertex is the Point A, and Baſe BCLE.

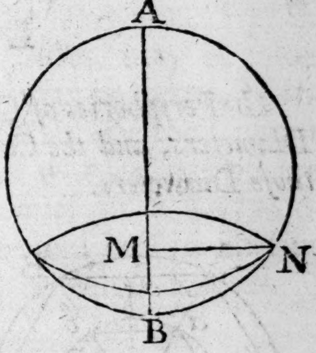


5. A Cylinder (DCcd) is a Solid generated by the Rotation of a Rectangle ACDB about one of its Sides AB ſuppoſed at Reſt; which Side AB is call'd the Axis of the Cone Cylinder.



6. A Cone (ACc) is a Solid generated by the Rotation of a right-angled Triangle ABC about its Perpendicular AB call'd the Axis of the Cone.

7. A



7. A Sphere is a Solid generated by the Rotation of a Semicircle about its Diameter; as A B.

From the three laſt Definitions, *it appears, that, if a Cylinder, Cone, or Sphere, be cut by a Plane, perpendicular to the Axis, the Section will be a Circle; becauſe, it is manifeſt, from the Geneſis, that any Right-line M N in the generating Plane, perpendicular to the Axis AB, deſcribes a Circle.*

8. The Fruſtum of a Pyramid or Cone is that Part which remains, when any Part next the Vertex, cut off by a Plane parallel to the Baſe, is taken away.

9. The Altitude of any Solid is the perpendicular Height of the Top, or Vertex, above the Baſe thereof.

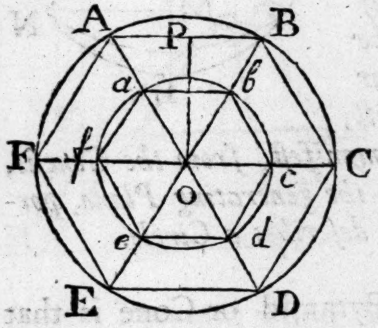
POSTULATUM.

If two Solids of the ſame Altitude be cut by Planes parallel to their Baſes, and the Sections, at all equal Diſtances from the Baſes, be either equal, or in a conſtant Ratio, the Solids themſelves will, accordingly, be either equal or in the ſame conſtant Ratio.

This will appear evident, if the Solids be conſider'd as compoſed of an equal Number of infinitely thin Slices; ſince it is eaſy to conceive and demonſtrate, that, if every Part or Slice of the one be, once, twice, or thrice, &c. as great as the corresponding Part or Slice of the other, the Whole of the one will alſo be once, twice, or thrice, as great as the Whole of the other.

LEMMA.

The Peripheries of Circles are, to each other, as their Diameters, and the Circles themſelves as the Squares of thoſe Diameters.



For let ABCDEF and abcdef be ſimilar equal-ſided Polygons inſcribed in the two propoſed Circles, and, from the common Center O, let the Right-lines OaA, ObB, *&c.* be drawn; then, becauſe of the ſimilar Triangles OAB and Oab, we ſhall have OA : Oa ($\because$ AB : ab) ſo is the Sum of all the Sides of the greater Polygon ABCDEF, to the Sum of all the Sides of the leſſer Polygon abcdef: Alſo, $OA^2 : Oa^2$ ($\because$ Triangle OAB: Triangle Oab) ſo is the greater Polygon, to the leſſer. Therefore, if the Number of Sides in each Polygon be now ſuppoſed indefinitely great, ſo that, inſtead of their Sum, their Limit, which is the Periphery of the Circle, may be taken, and, inſtead of the Polygon, the Circle itſelf; then, from the foregoing Proportions, we ſhall have OA : Oa :: the Periphery of the greater Circle : the Periphery of the leſſer; and $OA^2 : Oa^2$ :: the greater Circle : the leſſer. *Whence, the Truth of the Propoſition is manifeſt.*

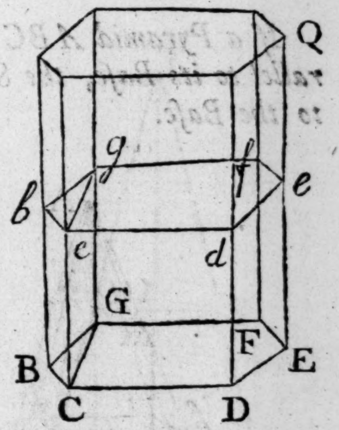
COROLLARY.

If OP be perpendicular to AB, then, the Triangle OAB being equal to AB × $\frac{1}{2}$ PO, the whole Polygon

Polygon ABCDEF is equal to the Sum of its Sides × $\frac{1}{2}$ PO: But when the Polygon, by increaſing the Number of its Sides, may be taken for the Circle itſelf, then PO may be taken for the Radius, and the Sum of the Sides, for the Circumference of the Circle: From whence it appears, that any Circle OACE is equal to a Rectangle under its whole Circumference and half its Radius, or half its Circumference and its whole Radius.

THEOREM I.

If a Priſm BQ be cut by a Plane parallel to the Baſe BCDEFG, the Section bcdefg will be equal to the Baſe.



Let the Points C, G and c, g be joined.

Since the Plane cf is parallel to the Plane CF, the Side bc will alſo be parallel to the Side BC, and therefore, bB being parallel to cC (by Def. 3.) BCcb is a Parallelogram, and conſequently [illegible] = BC. After the very ſame Manner, it will appear that, bg is = BG and cg = CG; whence, the Triangles bcg and BCG, being mutually equilateral, are alſo equal to each other: And, as the like may be inferred with regard to any other two correſponding Triangles, into which cf and CF may be divided, it is therefore evident that the whole Polygons are alſo equal. *Q. E. D.*

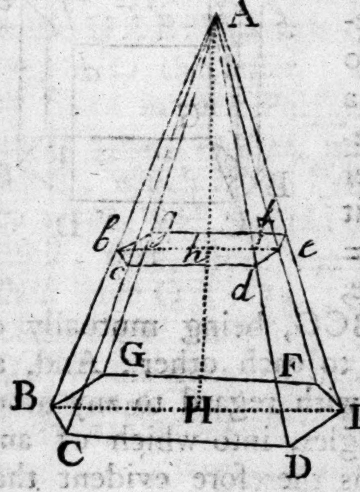
COROLLARY.

Becauſe any Section of a Cylinder parallel to the Baſe appears to be a Circle equal to the Baſe (from what has been already ſhewn at the End of Def. 7.); it therefore follows, that all Priſms and Cylinders, having equal Altitudes, have the ſame Ratio as their Baſes (*by the Poſtulatum*), their Sections being every where, in that conſtant Ratio. Whence, it is, alſo, manifeſt, that if the Baſes, as well as the Altitudes, be equal, the Solids themſelves will be equal.

THEOREM II.

If a Pyramid ABCDEFG be cut by a Plane parallel to its Baſe, the Section b c d e f g will be ſimilar to the Baſe.

For, becauſe of the parallel Lines b g, B G, and b c, BC, it will be, b c : BC ∷ A b : AB ∷ b g : B G; whence, alternately, b c : b g ∷ B C : B G. By the ſame Argument, it will appear, that b c : c d ∷ B C : C D; therefore, ſeeing the Sides about the equal, or correſponding, Angles of the two Figures are proportional, the Figures themſelves are ſimilar. *Q. E. D.*



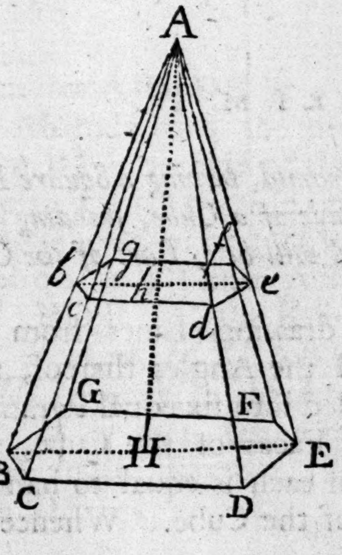
COROL-

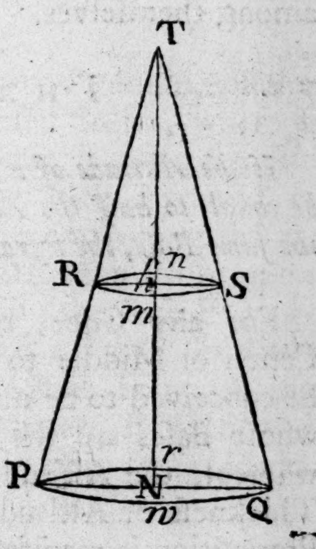
COROLLARY.

If AH, perpendicular to the Baſe CF, cut cf in h; then will $AH^2 : Ah^2 :: AB^2 :: Ab^2 :: BC^2 : bc^2 ::$ the Polygon CF : Polygon cf: Whence, it appears, that the Sections of a Pyramid, parallel to the Baſe, are to one another as the Squares of the Diſtances from the Vertex of the Pyramid.

THEOREM III.

Pyramids and Cones, whoſe Altitudes are equal, are to one another as their Baſes.





For, let bcdefg and RmSn be any two Sections of the propoſed Solids, parallel to, and equally diſtant from, their Baſes BCDEFG and PwQr: Then, becauſe the Altitudes AH and TN are equal, the Diſtances Ah, Th of thoſe Sections, from the Vertices A and F, will alſo be equal.

equal. But (*by Corol. to the precedent*) $AH^2 : Ah^2 :: BCDEFG : bcdefg$: And (*by the foregoing Lemma*) $TN^2\ (AH^2) : Th^2\ (Ah^2) :: (PN^2 : Rh^2 ::)\ PwQr : RmSn$; therefore, by Equality, $BCDEFG : bcdefg :: PwQr : RmSn$; or, alternately, $BCDEFG : PwQr :: bcdefg : RmSn$. Therefore, the Sections $bcdefg$, $RmSn$ being, every where, in the conſtant Ratio of the Baſes, it is evident, from what was premiſed in Page 121, that the Solids themſelves muſt, alſo, be in that Ratio. *Q. E. D.*

COROLLARY.

Hence, it follows that all Cones and Pyramids, whoſe Baſes and Altitudes are equal, are equal among themſelves.

THEOREM IV.

If the Altitude of a Pyramid, having a Square Baſe, be equal to half the Altitude of a Cube, ſtanding upon the ſame Baſe, the Pyramid will be $\frac{1}{6}$ Part of the Cube.

For any Cube, by drawing Lines from the Center or Middle to all the Angles thereof, may be conceived to be divided into ſix equal Pyramids, whoſe Baſes are the ſix Faces of the Cube, and, whereof, the Altitude of each is equal to half the Thickneſs or Altitude of the Cube. Whence the Propoſition is manifeſt.

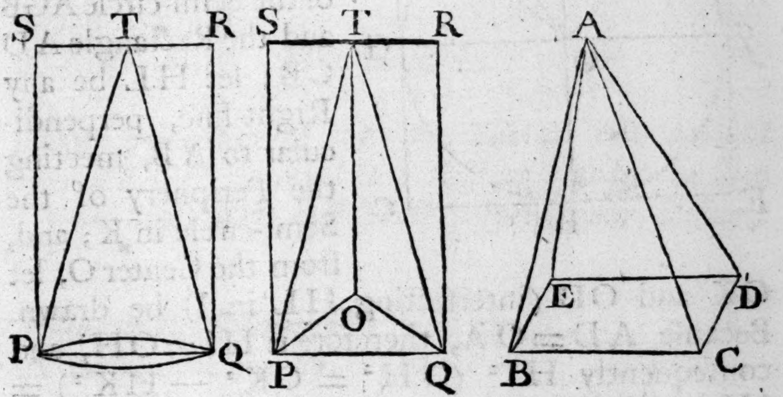
COROLLARY.

Hence, a Pyramid, having a Square for its Baſe, and its Altitude equal to half the Side of that Square, is $\frac{1}{3}$ of its circumſcribing Priſm, or $\frac{1}{3}$ of a Priſm having the ſame Baſe and Altitude.

THEO-

THEOREM V.

Every Pyramid or Cone PTQ is $\frac{1}{3}$ of its circumſcribing Priſm or Cylinder SPQR.

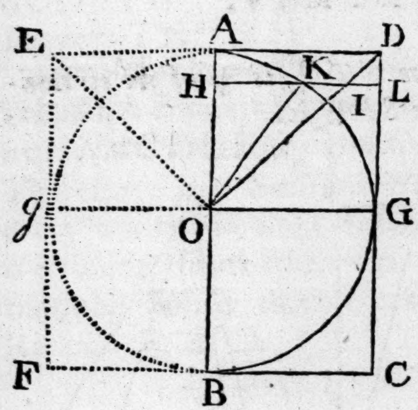


For, let ABCDE be a ſquare Pyramid, of the ſame Altitude with the propoſed Solids, whereof BC (or BE) the Side of the Baſe, is twice the Altitude: Then, becauſe every ſuch Pyramid is proved, above, to be $\frac{1}{3}$ of its circumſcribing Priſm, it will be, as 3 Times ABCDE (the Priſm upon BD) : PQRS :: the Baſe BD : the Baſe PQ (*by Corol. to Theor. I.*) :: ABCDE : TPQ (*by Theorem III.*); whence, the Antecedents being in the Ratio of 3 to 1, the Conſequents PQRS and TPQ muſt, likewiſe, be in the ſame Ratio. *Q. E. D.*

THEOREM VI.

Every Sphere is equal to $\frac{2}{3}$ of its circumſcribing Cylinder.

Let



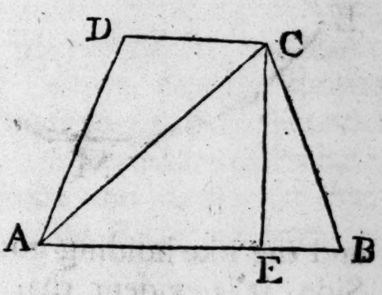
Let AB be the Axis about which the Sphere and Cylinder are generated, by the Rotation of the Semi-circle AGB and the Rectangle AD CB; let HL be any Right-line, perpendicular to AB, meeting the Periphery of the Semi-circle in K; and, from the Center O, let OK and OD (interſecting HL in I) be drawn. Becauſe $AD = OA$, therefore is $HI = OH$, and conſequently HI^2 $(OH^2 = OK^2 - HK^2) =$ $HL^2 - HK^2$; whence, becauſe all Circles are as the Square of their Radii (*by the foregoing Lemma*), it is evident, that the Circle deſcribed by HI, or the Section of the Cone, generated by the Triangle AOD, is equal to the Difference of the Circles deſcribed by HL and HK; that is, equal to the Annulus deſcribed by KL, or the Section of the Solid which remains when the Sphere is taken out of the Cylinder: Wherefore, ſeeing the Sections are every where equal, the Solids themſelves muſt alſo be equal, or the Cone EOD, generated by the Triangle AOD, equal to the Exceſs of the Cylinder GDEg above the Hemiſphere GAg; whence, as the Cone or Exceſs is $\frac{1}{3}$ of the Cylinder (*by the precedent*), the Hemiſphere muſt, conſequently, be the other two Thirds, and ſo, the whole Sphere equal to $\frac{2}{3}$ of its circumſcribing Cylinder CDEF. *Q. E. D.*

COROLLARY.

Hence, it appears, that a Cone, Hemiſphere, and Cylinder, having the ſame Baſe and Altitude, are to each other as 1, 2 and 3, reſpectively.

LEMMA.

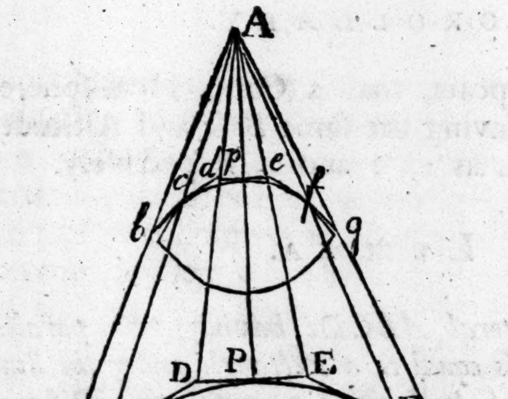
Any Quadrilateral ABCD, having two parallel Sides AB, CD, is equal to a Rectangle under the Sum of thoſe Sides and half their perpendicular Diſtance CE.



For, if the Diagonal AC be drawn, the Triangle ACB will be equal to $\frac{1}{2}CE \times AB$, and the Triangle $DAC = \frac{1}{2}CE \times CD$; and, therefore, the whole Quadrilateral $ABCD = \frac{1}{2}CE \times AB + \frac{1}{2}CE \times CD = \frac{1}{2}CE \times \overline{AB + CD}$. *Q. E. D.*

THEOREM VII.

The upper Superficies (or the Sum of all the Sides) of a regular Pyramid, in which a Cone may be inſcribed, is equal to a Rectangle under the Perimeter of its Baſe and half the ſlant Side of the Cone.



For, let BCDE, *&c.* be the Baſe of the Pyramid, and BPGM that of the inſcribed Cone; and, from the Point P, where any Side DE of the Polyg. touches the Circle, let the ſlant Side AP (equal to AB) be drawn: Then, the Triangle (or Side of the Pyramid) ADE being $= \frac{1}{2} AP \times DE$ or $\frac{1}{2} AB \times DE$, and the like holding good with regard to any other Side, it is evident that the Sum of all the Sides, or the upper Superficies of the whole Pyramid, will be equal to $\frac{1}{2} AB \times \overline{DE + EF + \&c}$; that is, equal to a Rectangle under $\frac{1}{2}$ AB and the whole Perimeter of the Baſe. *Q. E. D.*

COROLLARY I.

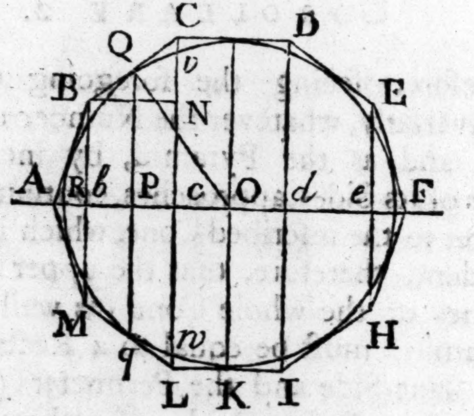
Hence, it will, alſo, appear, that all the Sides, of any Fruſtum Bg of the Pyramid, will be equal to a Rectangle under half the Length of each Side and the Sum of the Perimeters of the two Ends: For, the Area of the Side DEed being $= \frac{1}{2} Pp \times \overline{DE + de}$, or $\frac{1}{2} Bb \times \overline{DE + de}$ (*by the preceding Lemma*), the Area of all the Sides will, therefore, be $= \frac{1}{2} Bb \times \overline{DE + de + EF + ef + \&c.}$

COROLLARY 2.

Therefore, ſeeing the foregoing Concluſions hold univerſally, whatever the Number of the Sides may be, and as the Pyramid, by increaſing the Number of its Sides, approaches, continually, nearer and nearer to the inſcribed Cone, which is its Limit; it is evident, therefore, that the upper (or convex) Superficies of the whole Cone (as well as that of the Pyramid) muſt be equal to a Rectangle under half its ſlant Side and the Perimeter (or Circumference) of its Baſe: And, alſo, that the convex Superficies, of any Fruſtum of the Cone B g, is equal to a Rectangle under half its ſlant Side (B b) and the Sum of the Peripheries of its two Ends. From whence, it likewiſe follows, that the convex Surface of a Cylinder is equal to a Rectangle under half its Altitude and twice the Periphery of its Baſe, or under the whole Altitude and once that Periphery; becauſe then the two Ends are equal to each other.

THEOREM VIII.

If a regular Polygon ABCDE, &c. of an even Number of Sides, together with its inſcribed Circle RQSq, be ſuppoſed to revolve about the produced Diameter AF, as an Axis; the Superficies of the Solid, generated by the Polygon, will be equal to a Rectangle under its Axis AF and the Circumference RQSq of the inſcribed Circle.



From the Center O, to the Point of Contact Q, of any Side BC, let the Radius OQ be drawn; alſo draw BbM, QPq, CcL, *&c.* perpendicular to AF, and BN perpendicular to CL.

Becauſe the Solid, generated by the Plane *bc* CB, is the Fruſtum of a Cone, the convex Superficies thereof, generated by BC, is equal to a Rectangle under $\frac{1}{2}$ BC and the Sum of the two Peripheries deſcribed by Bb and Cc (*by Cor.* 2. *to the precedent*): But the Sum of theſe two, as Qq is an arithmetical Mean between them, is equal to twice the Periphery Qq; and, therefore, the convex Superficies of the ſaid Fruſtum is equal to $\frac{1}{2}$ BC × 2 Periph. Qq = BC × Periph. Qq. But, becauſe of the ſimilar Triangles OPQ, BNC, we have BC : BN (bc) :: OQ : PQ :: Periph. RQSq : Periph. Qq (*by the Lemma in Page* 122); and conſequently BC × Periph. Qq = bc × Periph. RQSq = the Superficies generated by BC. By the very ſame Argument, it will appear, that, the Superficies generated by any other Side CD is = cO × Periph. RQSq: Whence, it is manifeſt, that, the Superficies of the whole Solid is = $\overline{\text{Ab}+\text{bc}+\text{cO}+\&c.}\times$ Periph. RQSq. *Q. E. D.*

COROL-

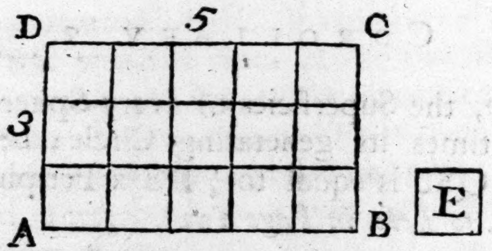
COROLLARY 1.

Since the whole Superficies is always equal to AF × Periph. RQSq, let the Number of Sides of the generating Polygon be what it will, and as the ſaid Superficies, by increaſing the Number of its Sides, approaches nearer and nearer, continually, to the Superficies of the inſcribed Sphere, which is its Limit; it is evident, that the Superficies of the Sphere itſelf is likewiſe equal to a Rectangle under its Axis RS and Periphery RQSq; and that the convex Superficies of any Segment thereof, vRw, is equal to a Rectangle under its Height Rc and the Circumference of the Sphere; becauſe the Superficies of the correſponding Part of the Solid, CBAML, is equal to Ac × Periph. RQSp, *as has been already proved.*

COROLLARY 2.

Hence, the Superficies of every Sphere is equal to four times its generating Circle; becauſe the Circle RQSq is equal to $\frac{1}{4}$ RS × Periph. RQSq, (*by Corol. to Lem. in Page* 122).

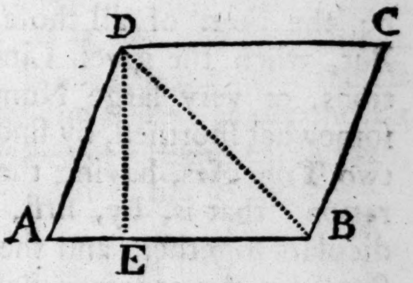
OF THE
MENSURATION
OF
SUPERFICIES and SOLIDS.



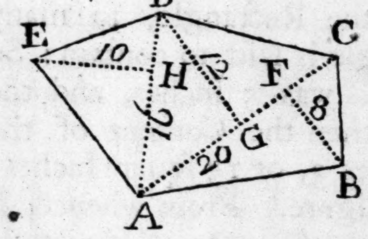
EVERY Quantity is meaſured by ſome other Quantity of the ſame Kind; as a Line by a Line, a Surface by a Surface, and a Solid by a Solid: And the Number which ſhews how often the leſſer, called the meaſuring Unit, is contained in the greater, or Quantity meaſured, is call'd the Content of the Quantity ſo meaſured. Thus, if the Quantity to be meaſured be the Rectangle ABCD, and the little Square E, whoſe Side

is

is one Inch (Foot, Yard, *&c.*) be the meaſuring Unit propounded; then, as often as the ſaid little Square is contained in the Rectangle, ſo many ſquare Inches the Rectangle is ſaid to contain: So that, if the Length AC was 5 Inches, and the Breadth AD 3 Inches, then the Content of the Rectangle would be 5 times 3, or 15 ſquare Inches; as will appear from the Figure. From whence, it is eaſy to conceive, that the Content of any Rectangle, whatever, will be found by repeating the Number of Parts into which the Length is divided (by the Side of the meaſuring Unit) as often as there are like Parts in the Breadth of the ſame Rectangle. Therefore, *to find the Content or Area of any Rectangle; Multiply the Length by the Breadth, and the Product will be the Anſwer.* Thus, let the Length be 18 Inches, and the Breadth 15, then the Content will be 15 times 18, or 270 ſquare Inches.



The Method of finding the Content of a Rectangle being thus known, the Manner of finding the Content of any Parallelogram ABCD, or Triangle ABD, is alſo known; the former of theſe Figures being equal to a Rectangle of the ſame Baſe and Altitude, and the latter equal to the half of ſuch a Rectangle (by Cor. 2. to 2. 2) Therefore, *Multiply the Baſe by the Perpendicular, for the Content of any Rectangle; and the Baſe by half the Perpendicular, for that of any Triangle.* Thus, for Example, let the Baſe be 18 Parts, and the Perpendicular 12 Parts; then the Content of the Parallelogram will be 216, and that of the Triangle 108 ſquare Parts.



From the Manner of finding the Area of a Triangle, the Area of any right-lined plane Figure, as ABCDE, may be determined, by dividing the whole into Triangles and finding the Content of each Triangle. Thus, let the dividing Lines AC, and AD, be 20 and 16 Inches, and the Perpendiculars BF, DG, EH, falling thereon, 8, 12 and 16 reſpectively; then, the Content of the

$$\text{Triangle} \left\{\begin{matrix} ABC \\ ACD \\ ADE \end{matrix}\right\} \text{being} \left\{\begin{matrix} 80 \\ 120 \\ 80 \end{matrix}\right\}, \text{it is e-}$$

vident, that the Content of the whole Figure will be the Sum of all thoſe, or 280 ſquare Inches. But, when the given Lines are expreſſed by Fractions, or very large Numbers, the Work will be ſomewhat ſhortned, by finding the Content of every two Triangles, having the ſame Baſe, at one Operation; that is, by, firſt, adding the two Perpendiculars together, and then multiplying half their Sum by the common Baſe of the two Triangles. Thus, in the laſt Example, the Half-ſum of the two Perpendiculars BF and DG being 10, if this Number be, therefore, multiplied by 20 the Meaſure of the common Baſe AC, the Product, which is 200, will be the Content of the Trapezium ABCDA, to which 80 the Content of the Triangle ADE being added, the Sum will be 280, the ſame as before. But, if the Polygon propoſed be a regular one, that is, one whoſe Sides and Angles

gles are all equal, the ſhorteſt Way, of all, is, to *multiply half the Sum of its Sides by the Length of the Line drawn from the middle of any Side to the Center of the Polygon:* The Reaſon of which is obvious, from the Demonſtration to the Lemma in Page 122.

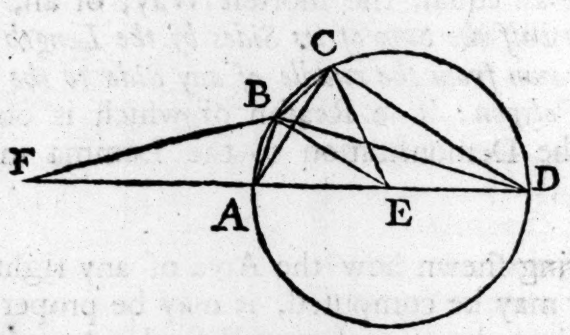
Having ſhewn how the Area of any right-lined Figure may be computed, it may be proper to ſay ſomething, here, with regard to the Area and Periphery of the Circle.

It is well known, that to determine the true Area of a Circle, and to find a Right-line exactly equal to the Circumference thereof, have been looked upon, by Mathematicians, as abſolutely impoſſible: But, tho' neither the one nor the other can be accurately known, yet ſeveral Ways have been invented, by which they may be approximated to any aſſigned Degree of Exactneſs; whereof, that which I am going to lay down, tho' it is not the moſt ſhort, and expeditious, of all others, ſeems nevertheleſs to be the moſt proper for this Place, as depending on the moſt ſimple and evident Principles: I ſhall, therefore, begin with premiſing the following

LEMMA.

If AD be a Diameter, and AB, BC two equal Arcs of the ſame Circle, and DB, DC be the Chords of the Supplements of thoſe Arcs; then will $DB^2 = \frac{1}{2}AD \times DC + \frac{1}{2}AD^2$.

For,



For, in DA, produced, take $AF = DC$, and draw BF, BA, BC, and the Radius BE: Then, ſince the external Angle FAB, of the Trapezium $ABCD$, is equal to the internal oppoſite Angle DCB (*by Cor.* 1. *to* 13. 3.) alſo $AF = DC$, and $AB = CB$ (*by Hyp.*), it is evident, that, FB is alſo $= DB$, and conſequently the Angle $F = FDB = DBE$: Whence, the iſoſceles Triangles DEB, DBF being equiangular, it will be $DE\ (\frac{1}{2} AD) : DB :: DB : DF\ (DC + AD)$, and conſequently $DB^2 = \frac{1}{2} AD \times DC + \frac{1}{2} AD^2$. *Q. E. D.*

COROLLARY.

Hence, if the Diameter AD be denoted by the Number 2, the Chord DB will be denoted by $\sqrt{DC + 2}$: Whence, it appears, that, *if the Meaſure of the Supplement-chord of any Arch be increaſed by the Number* 2, *the Square-root of the Sum will be the Supplement-chord of half that Arch.*

Now to apply this to the Matter propoſed, that is, to the finding the Area and Circumference of the Circle, let the Arch ABC be taken equal to $\frac{1}{3}$ of the Semi-periphery ACD, or the Angle AEC equal to $\frac{1}{3}$ of two Right-angles; then the Triangle AEC being equiangular (*by Cor.* 7. *to* 10. 1.), the Chord AC will alſo be equal to the Radius AE or

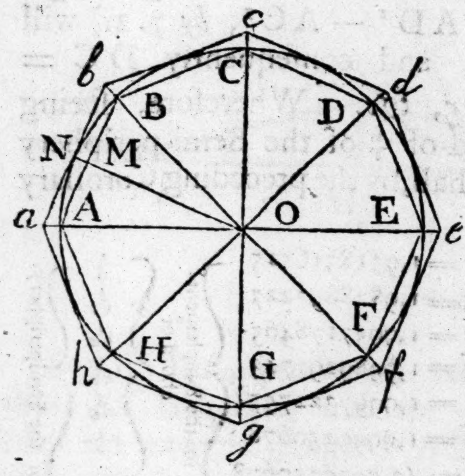
EC;

EC; and therefore, ſince ACD is a Right-angle (*by* 11. 3,) DC^2 ($= AD^2 - AC^2$, *by* 7. 2.) will be $= 4 - 1 = 3$, and conſequently $DC = \sqrt{3} = 1{,}7320508075$, *&c.* Wherefore, ſeeing the Supplement-chord of $\frac{1}{3}$ of the Semi-periphery is 1,7320508075, we ſhall, by the preceding Corollary

$$\text{have}\left\{\begin{array}{l} \sqrt{2+1{,}7320508075}=1{,}9318516525 \\ \sqrt{2+1{,}9318516525}=1{,}9828897227 \\ \sqrt{2+1{,}9828897227}=1{,}9957178465 \\ \sqrt{2+1{,}9957178465}=1{,}9989291743 \\ \sqrt{2+1{,}9989291743}=1{,}9997322757 \\ \sqrt{2+1{,}9997322757}=1{,}9999330678 \\ \sqrt{2+1{,}9999330678}=\sqrt{3{,}9999330678} \end{array}\right\}\text{for the Supplement-chord of}\left\{\begin{array}{c} \frac{1}{6} \\ \frac{1}{12} \\ \frac{1}{24} \\ \frac{1}{48} \\ \frac{1}{96} \\ \frac{1}{192} \\ \frac{1}{384} \end{array}\right\}\text{of the Semiperiphery.}$$

Now, therefore, ſince it is found that 39999330678 is the Square of the Supplement-chord of $\frac{1}{384}$ of the Semi-periphery, let this Number be ſubſtracted from 4 the Square of the Diameter, and the Remainder 0,0000669322 will be the Square of the Chord of the ſame Arch; therefore, the Chord itſelf being $= \sqrt{0{,}0000669322} = 0{,}00818121$, let this Number be multiplied by 768 (or twice 384) and the Product 6,28317 will be the Perimeter of a regular Polygon of 768 Sides inſcribed in the Circle; which, as the Sides of the Polygon very nearly coincide with the Circumference of the Circle, muſt alſo expreſs the Length of the Circumference itſelf very nearly.

But,



But, in order to ſhew how near this is to Truth, let AB repreſent the Side of a Polygon of 768 Sides inſcribed in the foreſaid Circle, whoſe Length we have found above to be 0,00818121; alſo, let *ab* be the Side of another Polygon, of the ſame Number of Sides, deſcribed about the Circle, and from the Center O, let ON be drawn, biſecting AB and *ab* in M and N: Then, ſince AM is $= \frac{1}{2}$ AB $=$ 0,0040906, and AO $=$ 1, it is plain OM^2 ($AO^2 - AM^2$) will be $=$ 0,99998327, and conſequently OM $=$ 0,99999163; whence, becauſe of the ſimilar Triangles AOB, aOb, *&c.* we have 0,999991632 (OM): 1 (ON) :: AB : *ab* :: 6,28317 (the Perimeter of the inſcribed Polygon) : 6,28322 the Perimeter of the circumſcribed Polygon. Now, ſince the Circumference of the Circle is greater than the Perimeter of the inſcribed Polygon, and leſs than that of the circumſcribed, it muſt, therefore, be greater than 6,28317, and leſs than 6,28322; and therefore is equal to 6,2832, very nearly, ſince this Number exceeds the Perimeter of the inſcribed Polygon by no more than 0,00003, and is exceeded by the Perimeter of the circumſcribed one, by 0,00002 only.

From the Periphery thus found, the Area of the Circle will alſo be known, being equal to the Rectangle of the whole Periphery and half the Radius (*by Cor. to Lemma in Page* 122.) that is $= 6{,}2832 \times 0{,}5$ or 3,1416.

There-

Therefore, ſince it is proved in the ſame Lemma that the Peripheries of Circles are as their Diameters, and the Circles themſelves as the Squares of thoſe Diameters, it follows that, as 2 to 6,2832, or as 1 to 3,1416 ∷ the Diameter of any Circle to its Periphery; and as 4 to 3,1416, or as 1 to 0,7854 ∷ the Square of the Diameter to the Area, very nearly.

But, if you had rather have the Proportions in whole Numbers, and the Caſe propoſed does not require any great Degree of Accuracy; then, inſtead of the foregoing, thoſe of *Archimedes* may be uſed, *viz.* 7 : 22 ∷ Diam. : Circumf. and 14 : 11 ∷ Square Diam. : Area; which Proportions differ but little from thoſe above, as will appear from the following Example; wherein the Diameter of a Circle being 28, its Circumference and Area are required: Here, according to the firſt Proportions, I multiply 28 by 3,1416 for the Circumference, and the Square of 28 (or 784) by 0,7854 for the Area, and there reſults 87,964, and 615,75 reſpectively: But, according to the Proportions of *Archimedes*, the Circumference will be found equal to 88 and the Area 616; which differ very little from the former. After the ſame Manner the Circumference or Area of any other Circle may be determined.

From the Manner of computing the Areas of plain Figures (already explained), the convex Superficies of ſolid Bodies may alſo be determined. Thus,

The convex Superficies of a Cylinder is found, by *multiplying the Periphery of the Baſe by the Altitude of the Solid* (as appears from Cor. 2. to Theor. 7, Page 131): Therefore, if, to the Product, the Area of of the two Ends (*found by the preceding Rules*) be added, the Sum will be the whole Superficies of the Cylinder.

To find the convex Superficies of a Cone; *Multiply half the ſlant Side thereof by the Circumference of its Baſe.*

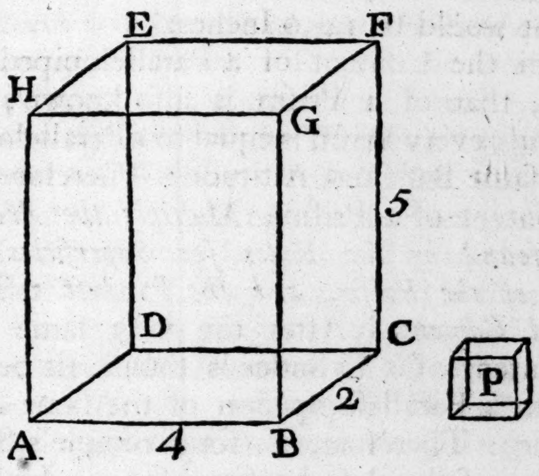
The convex Superficies of any Fruſtum of this Solid is found, by *multiplying the Sum of the Peripheries of the two Ends by half the Length of the ſlant Side of the Fruſtum.*

The Reaſon of theſe two laſt Rules is demonſtrated in Cor. 2 to Theor. 7. Page 131.

To find the Superficies of a Sphere; *Multiply the Periphery of the greateſt, or generating, Circle by its Diameter*; *or, multiply the Square of the Diameter by* 3,1416. The Reaſon of which will appear from Cor. 1. to Theor. 8; whence the Rule for determining the convex Superficies of any Segment of a Sphere will, alſo, appear evident, and is thus; *Multiply the Periphery of the greateſt Circle of the Sphere by the Altitude of the Segment.*

Having explained how the Superficies of ſolid Bodies are determined, it remains, next, to ſhew how the Contents of the Solids themſelves may be found. And here, it will be, firſt, neceſſary to obſerve, that, as every Superficies is meaſured by a Square whoſe Side is Unity, ſo every Solid is meaſured by a Cube, whereof the Length of each Side is alſo an Unit.

Let



Let A F be a ſolid Parallelopipedon, to be meaſured by the Cube P, whoſe Side is one ſquare Inch; moreover, let the Length AB of the Baſe be 4 Inches, the Breadth, AD, two Inches, and the Height AH, or DE, of the Solid, 5 Inches: Then, becauſe the Area of the Baſe ABCD is 4×2 or 8 ſquare Inches, it is eaſy to conceive, that, if the Solid was only one Inch high (inſtead of 5) the Content thereof would be juſt the ſame Number (8) of cubical Inches; becauſe then, upon each of the 8 Squares of the Baſe ABCD, a Cube, whoſe Side is one Inch, might be erected; which 8 Cubes together would, it is manifeſt, compoſe a Parallelopipedon, upon the ſame Baſe, of one Inch high. Therefore, ſeeing the Content of the Solid, at one Inch high, is 8 cubical Inches, the Content, at 5 Inches high, muſt, conſequently, be equal to 5 times 8 or 40 cubic Inches (becauſe the whole Solid may be conſider'd as compoſed of 5 ſuch Heights of Cubes one above another); whence, it is manifeſt that, whatever the Height may be, the Content will be always found by *multiplying the Area of the Baſe by the Altitude of the Parallelopipedon:* Thus, if the Area of the Baſe was the Baſe was 725, and the Altitude 2 Inches, the Content would be 1450 Inches.

From the Content of a Parallelopipedon thus known, that of a Priſm is alſo known; becauſe (*by Poſtul.*) every Priſm is equal to a Parallelopipedon of the ſame Baſe and Altitude: Therefore, to find the Content of a Priſm; *Multiply the Area of the Baſe* (*found by the Rules for Superficies*) *by the Height of the Priſm, and the Product will be the required Content.* After the very ſame Manner the Content of a Cylinder is found (it being alſo equal to a Parallelopipedon of the ſame Baſe and Altitude): Therefore, if (for Example's Sake) the Diameter of the Baſe be ſuppoſed 20 Inches, and the Height of the Cylinder 15 Inches, then the Content of the Solid will be 4712,4 cubical Inches, very nearly. For, the Area of the circular Baſe being 314,16 (*by Page* 140), this multiplied by 15 gives 4712,4 as before.

Hence, alſo, the Solidity of a Pyramid, or Cone, is likewiſe known; every ſuch Solid being $\frac{1}{3}$ of a Priſm or Cylinder of the ſame Baſe and Altitude: Therefore, to find the Content of a Pyramid or Cone; *Multiply the Area of the Baſe by* $\frac{1}{3}$ *of the Altitude.*

To find the Content of any Fruſtum of either of theſe two laſt Solids; firſt find, the Content of the whole Solid, whereof it is a Fruſtum, and afterwards of the Part cut off (by the ſame Rule), and then ſubſtract the latter from the former.

Every Sphere being (*by Theor.* 6. *Page* 127) $\frac{2}{3}$ of a Cylinder of the ſame Diameter and Altitude, the Content of a Sphere is, therefore, found, by *multiplying the Area of its greateſt, or generating, Circle by* $\frac{2}{3}$ *of its Diameter*; or, becauſe the Area of ſuch Circle is to the Square of the Diameter, as 0,7854 to to Unity; *Multiply the Cube of the Diameter by .5236 which is* $\frac{2}{3}$ *of* 0,7854, *and the Product will be the Content of the Sphere.* Thus if the Diameter of a Sphere be 20, the Cube thereof will be 8000, which multiplied by the Fraction .5236 gives 4188,8 for the Solidity of the Sphere very nearly.

The Manner of finding the Content of a Segment of a Sphere may be gather'd from the *Demonſtration in Theor. VI. Page* 127, and is thus; *Multiply the Square of the Height or Thickneſs of the Fruſtum by* 3 *times the Diameter of the Sphere leſs by twice the ſaid Height, multiply the Product again by the Fraction* .5236, *and the Product will be the Content of the Segment very nearly.*

THE CONSTRUCTION

Of a Variety of

GEOMETRICAL PROBLEMS,

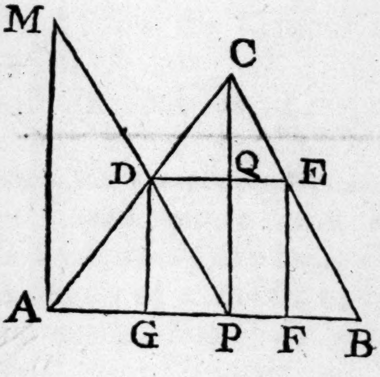
BEING

An APPLICATION of what has been deliver'd in ſome of the preceding Pages.

PROBLEM I.

In a given Triangle ABC, to inſcribe a Square DEFG.

CONSTRUCTION.



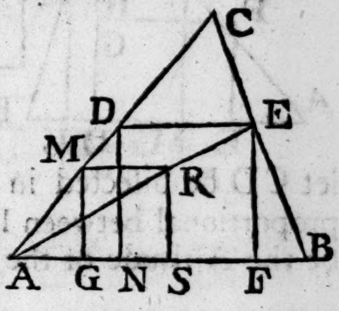
UPON the Baſe AB, from the Vertex C, let fall the Perpendicular CP, and make AM parallel to PC, and equal to AB; alſo draw PM cutting the Side AC in D, and draw DE, parallel to AB, meeting BC in E; then upon AB let fall the Perpendiculars DG, EF, *and the Thing is done.*

DEMON-

DEMONSTRATION.

It is evident, from the Conſtruction, that the Figure DEFG is a Rectangle. And, by ſimilar Triangles, AP : AM (AB) :: PG (QD) : DG; alſo (*by* 12. 4.) AP : AB :: DQ : DE; therefore DG is = DE; and conſequently DEFG a Square. *Q. E. D.*

The ſame conſtructed otherwiſe.



From any Point M, in either Side, upon the Baſe AB, let fall the Perpendicular MG, and make MR perpendicular, and equal, thereto; and let ARE be drawn meeting the other Side of the Triangle in E; then draw ED parallel, and EF and DN perpendicular, to AB, for the Sides of the required Square.

DEMONSTRATION.

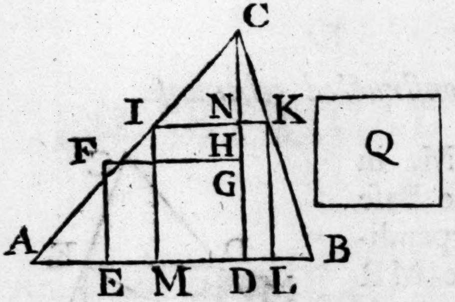
Let RS, parallel to EF, be drawn; and then, by ſimilar Triangles, RS (MG) : EF (:: AR : AE) :: MR : DE; therefore, MG and MR being equal, by Conſtruction, EF and DE will alſo be equal. *Q. E. D.*

Note. By either of theſe Methods a Rectangle may be inſcribed in a Triangle, whoſe Sides ſhall be in a given Ratio, if, inſtead of taking AM = AB, in the firſt Method, and MR = MG, in the ſecond, they be taken in the given Ratio; the reſt of the Conſtruction being exactly the ſame.

PROBLEM II.

In a given Triangle ABC, to inſcribe a Rectangle IKLM equal to a given right-lined Figure Q.

CONSTRUCTION.



Upon the Baſe AB having let fall the Perpendicular CD, take $ED = \frac{1}{2} AB$, upon which let a Rectangle DEFG be deſcribed equal to Q (*by* 7. 6); alſo let CD be biſected in H, and take HN a Mean-proportional between DH and HG; then will DN be the Altitude of the required Rectangle.

DEMONSTRATION.

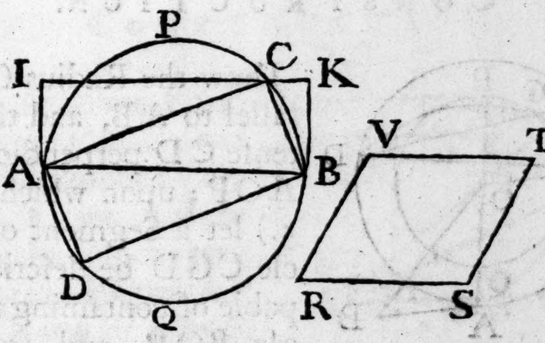
Becauſe $DH : HN :: HN : HG$ (*by Conſtr.*), therefore $HN^2 = DH \times HG$ (3. 4), and conſequently $DH^2 - HN^2 = DH^2 - \overline{DH \times HG}$; but $DH^2 - HN^2$ $(= \overline{DH + HN} \times \overline{DH - HN}) = DN \times CN$ (6. 2), and $DH^2 - DH \times HG = DH \times DG$ (4. 2.); whence, alſo, $DN \times CN = DH \times DG$, and conſequently $DN : DG :: DH : CN$ (3. 4); but DH ($\frac{1}{2}$ CD) $: CN :: ED$ ($\frac{1}{2}$ AB) $: IK$ (12. 4.); therefore, by Equality, $DN : DG :: ED : IK$, and conſequently $DN \times IK = DG \times ED$ (3. 4.) $=$ Q (*Conſtr.*). *Q. E. D.*

Note. This Problem is impoſſible when the Point G falls above H, or when the Figure to be inſcribed is greater than half the Triangle.

PROBLEM III.

In a given Circle APBQ, to inſcribe a Rectangle equal to a given right-lined Figure RSTU.

CONSTRUCTION.



Upon the Diameter AB deſcribe the Rectangle ABKI = RSTU (*by* 7. 6.), and from the Point C, where the Side KI interſects the Periphery of the Circle, draw CA and CB; parallel to which draw BD and AD; then will ACBD be the Rectangle that was to be conſtructed.

DEMONSTRATION.

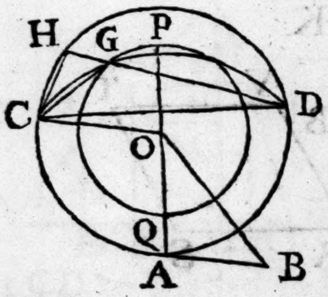
The Angle ACB, ſtanding in a Semi-circle, is a Right-angle (11. 3.); therefore AC and BD, as alſo AD and BC being Parallels, by Conſtruction, ACBD is a Rectangle (11. 3. *and Cor.* 2. *to* 23. 1.), and is, alſo, = 2 ACB (23. 1.) = ABKI (*Cor.* 2. *to* 2. 2.) = RSTU (*Conſtr.*). *Q. E. D.*

Note. When the Side IK, inſtead of cutting (or touching), falls quite above the Circle, the Problem will be impoſſible; that is, no Rectangle ſo great as RSTU can poſſibly be inſcribed in the Circle.

PROBLEM IV.

To inſcribe a Triangle, betwixt the Peripheries of two given concentric Circles ACD, QPG, which ſhall be ſimilar to a given Triangle OAB.

CONSTRUCTION.



Draw the Radius OC parallel to AB, and the Subtenſe CD perpendicular to AOP; upon which (*by* 17. 5.) let a Segment of a Circle CGD be deſcribed capable of containing the Angle BOP, and from the Point G, where it meets the leſſer Circle QPG, draw GC and GD, and let the latter of theſe be produced to meet the Periphery of the greater Circle in H; then join C, H, and CGH will be the Triangle required.

DEMONSTRATION.

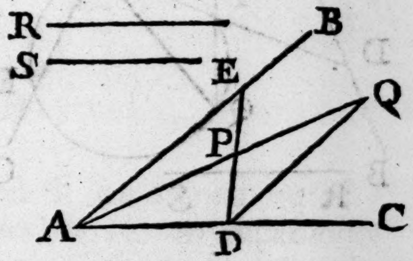
The Angles CGD and POB being equal, by Conſtruction, their Supplements CGH and AOB muſt likewiſe be equal; therefore, the Angle CHG being, alſo, equal to COA (9. 3.) = OAB (8. 1.), the Triangles CGH and OAB are ſimilar (*Cor. to* 5. 4.). *Q. E. D.*

This Problem is impoſſible when the Circle CGD neither cuts, nor touches the Circle PQ.

PROBLEM V.

Thro' a given Point P, to draw a Right-line DE, ſo that the Parts thereof PE, PD, intercepted by that Point and two Lines AB, AC, given by Poſition, may obtain a given Ratio.

CONSTRUCTION.



Let the given Ratio be that of R to S; and from the Point of Concourſe A, let AP be drawn, in which, produced, take PQ to AP in the Ratio of S to R (*by* 13. 5.); draw QD, parallel to BA, meeting AC in D; and draw DPE, *and the Thing is done.*

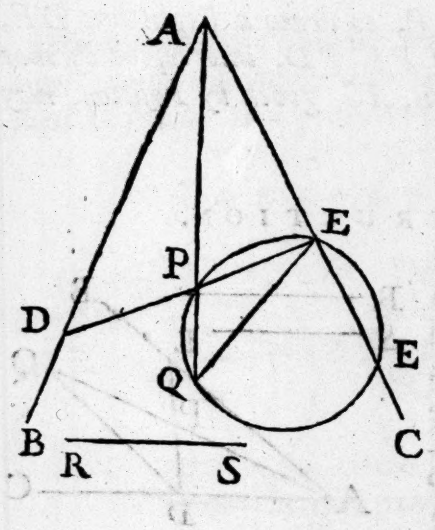
DEMONSTRATION.

Becauſe QD, is parallel to BA, the Triangles APE, QPD are equiangular (8. 1); and therefore EP : PD :: AP : PQ (5. 4.) :: R : S (*Conſtr.*). *Q. E. D.*

PROBLEM VI.

Thro' a given Point P, to draw a Right-line DE, terminating in two other Right-lines AB, AC, given by Poſition, ſo that the Rectangle under the Parts thereof PD, PE, intercepted by that Point and thoſe Lines, may be equal to a given Square RS^2.

CONSTRUCTION.



From the given Interſection A to the Point P draw AP, and in it produced take PQ a Third-proportional to AP and RS; upon which let a Segment of a Circle be deſcribed containing an Angle equal to BAP, and from either Point E, where it interſects AC, draw EPD, *and the Thing is done.*

DEMONSTRATION.

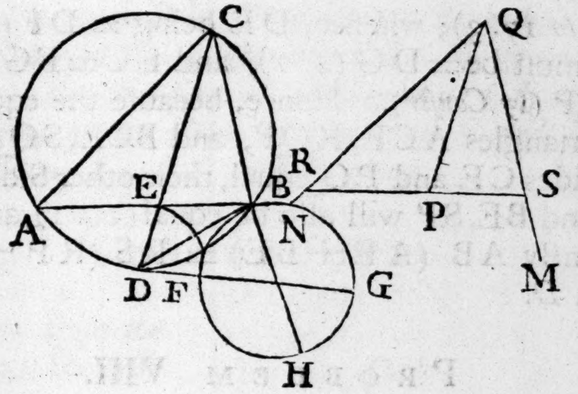
If QE be drawn, the Angle PEQ will be equal to DAP (BAP), *by Conſtr.*; and, therefore, EPQ being = APD, the Triangles APD, EPQ will be equiangular, and ſo AP : DP :: EP : QP, and conſequently AP × QP = DP × EP: But (*by Conſtr.*) AP : RS :: RS : QP; whence AP × QP is = RS^2, and conſequently DP × EP alſo = RS^2. *Q. E. D.*

PROBLEM VII.

Thro' a given Point P, equidiſtant from two Right-lines QM and QN, given by Poſition, to draw a Right-line RS, which, terminating in the two former, ſhall be equal to a given Right-line AB.

CONSTRUCTION.

CONSTRUCTION.



Upon the given Line A B, let a Segment of a Circle A C B be deſcribed (*by* 17. 5), capable of containing the given Angle N Q M, and let the whole Circle be completed; draw B D making the Angle ABD = NQP, and meeting the Circumference in D; make BH perpendicular to DB and equal to P Q, the Line drawn from P to the Point of Concourſe; and about BH, as a Diameter, let a Circle FHGB be deſcribed, and thro' the Center thereof draw DG, meeting the Periphery in F and G; upon the Center D, thro' F, deſcribe an Arch interſecting A B in E; draw DEC, and alſo RPS ſo as to make the Angle QPS = CEB, *and the Thing is done.*

DEMONSTRATION.

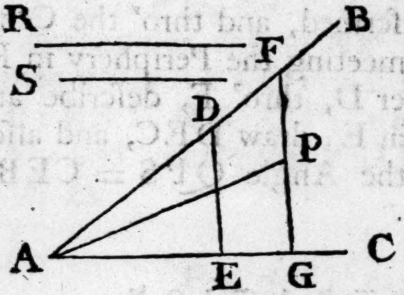
From the Point C, where DC interſects the Periphery, draw CA and CB: The Angle ABD being equal to half N Q M, or A C B (*by Conſtruction and Hypotheſis*), and ACD = ABD (*by Cor. to* 9. 3), it is evident that DCB (as well as ACD) is the half of ACB, and that the Triangles DCB, DBE, having

having D common and DCB = DBE, are ſimilar; therefore DC : DB :: DB : DE (5. 4), and conſequently DC × DE = DB² (3. 4.) = DG × DF (*Cor. to* 17. 3); whence, DE being = DF (*Conſtr*), DC muſt be = DG (4. 2), and EC = FG = BH = QP (*by Conſtr*). Hence, becauſe the equiangular Triangles ACE, RQP, and BCE, SQP have the Sides CE and PQ equal, their other Sides AE, RP and BE, SP will alſo be equal (20. 1), and conſequently AB (AE + BE) = RS (RP + SP). *Q. E. D.*

PROBLEM VIII.

Thro' a given Point P, betwixt two Right-lines AB, AC, given by Poſition, ſo to draw a Right-line FPG, that the Parts AF and AG of the two former, intercepted by that Line and the Point of Concourſe A, may obtain a given Ratio.

CONSTRUCTION.



Let the given Ratio be that of R to S, and in AB take AD = R, and in AC take AE = S; draw DE, and, parallel to DE, draw FPG, *and the Thing is done.*

DEMONSTRATION.

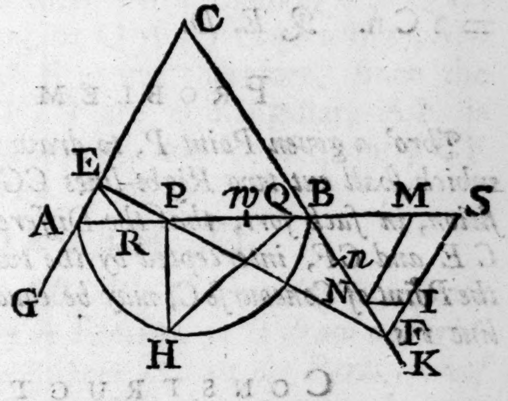
R : S (:: AD : AE, *by Conſtr.*) :: AF : AG (5. 4.). *Q. E. D.*

PROBLEM

PROBLEM IX.

Thro' a given Point P, to draw a Right-line EPF, which ſhall cut two Right-lines CG, CK, given by Poſition, ſo that the Sum of the Segments CE and CF, intercepted by the Point of Concourſe C and the two Interſections E and F, may be given.

CONSTRUCTION.



Thro' P (*by the precedent*) let AB be drawn, ſo as to cut off AC = CB; and in CK let Cn be taken equal to half the given Sum of the Segments CE and CF; make nN = nB, and draw NM, parallel to CG, meeting AB produced in M, and biſect AM in w: Upon AB let a Semicircle AHB be deſcribed, and let PH, perpendicular to AB, meet the Periphery thereof in H; and from the Center H, with a Radius equal to Aw (or Mw), let an Arch be deſcribed interſecting AB in Q; make QR = Aw and draw RE, parallel to CK, meeting CG in E; laſtly, draw EPF, *and the Thing is done.*

DEMONSTRATION.

Join H, Q and draw FS and NI parallel to CG and AM.

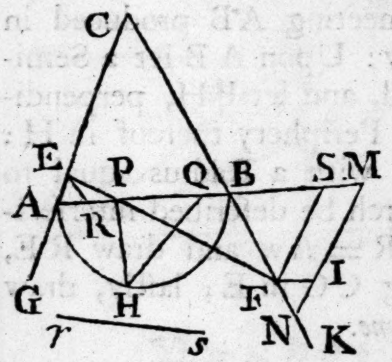
Then

Then $PR \times PS = PA \times PB$ (*by the precedent*) $= PH^2$ (*Cor. to* 11. 4.) $= QH^2\ (QR^2) - QP^2$ (*by Cor. to* 7. 2.) $= \overline{QR - QP} \times \overline{QR + QP}$ (*by* 6. 2.) $= PR \times \overline{QR + QP}$; whence, it is manifeſt that $PS = QR + QP$; and, conſequently, that $QS = QR = Aw$ (*by Conſtruction*): Therefore, ſince $RS\,(QS + QR) = 2\,Aw = AM$ (*by Conſtr.*); let RM, common, be taken away, and there will remain $MS = (NI) = AR$; whence, the Triangles NIF and ARE being equiangular, NF is alſo $= AE$, and conſequently $CE + CF = CA + CN = CA + CB + 2\,Bn = 2\,CB + 2\,Bn = 2\,Cn$. *Q. E. D.*

PROBLEM X.

Thro' a given Point P, to draw a Right-line EPF, which ſhall cut two Right-lines CG, CK, given by Poſition, in ſuch ſort, that the Difference of the Segments CE and CF, intercepted by the two Interſections and the Point of Concourſe C, may be equal to a given Right-line rs.

CONSTRUCTION.



Thro' P let AB be drawn, ſo as to cut off $CA = CB$ (*by Probl.* 8), and in BK take $BN = rs$; draw NM, parallel to GC, meeting AB, produced, in M, and biſect AM in Q: Upon AB let a Semi-circle AHB be deſcribed, and let PH be perpendicular to AB, and meet the Periphery in H; join Q, H, and in QA take $QR = QH$; draw RE, parallel to CK, meeting CG in E; laſtly draw EPF meeting CK in F, *and the Thing is done.*

DEMON-

DEMONSTRATION.

Draw FS parallel to MN, and FI parallel to AM. Becauſe of the parallel Lines AE, FS and ER, FB, it will be PR : PA :: PB : PS (*by* 12. 4), and therefore $PR \times PS = PA \times PB$ (*by* 3. 4) : But $PA \times PB = PH^2$ (*by Cor. to* 11. 4.) $= QH^2\ (QR^2) - QP^2$ (*by Cor. to* 7. 2.) $= \overline{QR - QP} \times \overline{QR + QP}$ (*by* 6. 2.) $= PR \times \overline{QR + QP}$; therefore $PR \times PS$ is, alſo, equal to $PR \times \overline{QR + QP}$, and conſequently $PS = QR + QP$; from which let PQ, common, be taken away, and there will remain $QS = QR$; whence, QM being = QA (*by Conſtruction*), SM (FI) will alſo = AR; and therefore, ſince the Triangles ARE, FIN are equiangular, AE is likewiſe = FN (*by* 20. 1), and conſequently CF — CE (= BN) = rs. *Q. E. D.*

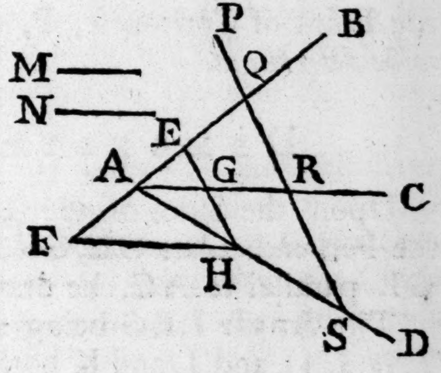
PROBLEM XI.

Three Right-lines AB, AC, AD, meeting in a given Point A, being given by Poſition, ſo to draw a Right-line PS, from a given Point P, that the Parts thereof QR, RS, intercepted by thoſe Lines, may obtain a given Ratio.

CONSTRUCTION.

Let the given Ratio of QR to RS be that of M to N:

In AB, produced, take AE = M, and AF = N, and draw FH, parallel to AC, meeting AD in H; alſo draw EH, and parallel to it draw PS, *and the Thing is done.*



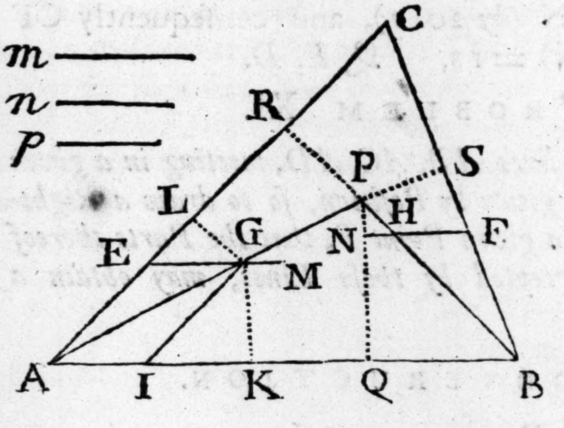
DEMONSTRATION.

Becauſe of the parallel Lines, it will be AE (M): AF (N)::EG:GH::QR:RS (*by* 4.4.). *Q. E. D.*

PROBLEM XII.

To find a Point P, from which three Perpendiculars, drawn to as many Right-lines AB, AC, BC, given by Poſition, may obtain the Ratio of three given Right-lines m, n, and p, reſpectively.

CONSTRUCTION.



Take AE and BF each = m, and draw EM and FN parallel to AB; in which take EG = n, and FH = p; thro' G and H draw AP and BP, and the Point of Concourſe, P, *will be that which was to be determined.*

DEMONSTRATION.

Upon the Sides of the Triangle ABC let fall the Perpendiculars GL, GK, PR, PQ, PS, and let GI, parallel to AC, be drawn.

The Angle LEG being = LAK = GIK (*Cor.* 1. *to* 8. 1), and L and K both Right-angles (*by Conſtruction*),

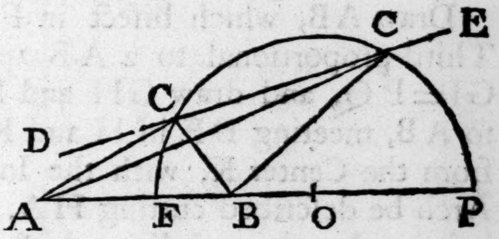
ſtruction), the Triangles EGL and GIK are ſimilar; and therefore IG (AE) : EG ∷ KG : LG (5. 4.); but KG : LG ∷ PQ : PR (13. 4); whence, by Equality, AE (m) : EG (n) ∷ PQ : PR. By the very ſame Method of Reaſoning, it will appear, that m : p ∷ QP : SP. *Q. E. D.*

After the very ſame Manner the Problem may be conſtructed, when the Lines drawn from P, inſtead of being perpendicular, make any given Angles with the Lines upon which they fall.

PROBLEM XIII.

From two given Points A, B, to draw two Right-lines AC, BC, which ſhall meet in a Right-line DE, given by Poſition, and alſo obtain a given Ratio.

CONSTRUCTION.



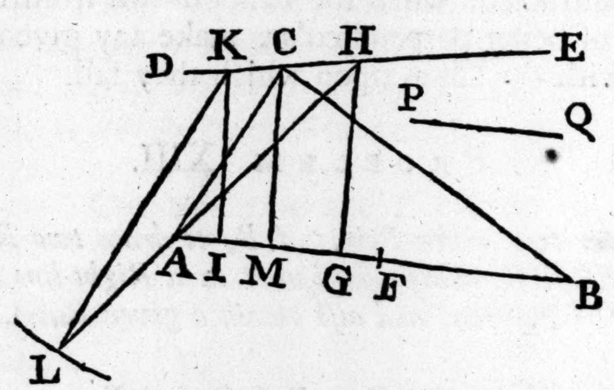
Thro' the two given Points draw the Right-line AP; and let AF be to BF in the Ratio given; make FO : AF ∷ BF : AF — BF, and, upon the Center O, with the Radius OF, let the Semicircle FCCP be deſcribed meeting the Right-line DE in C; join A, C and B, C, *and the Thing is done.*

The Demonſtration of which is manifeſt from Theorem XV. Book 4. From whence, it alſo appears, that, by the ſame Method, 2 Right-lines may be drawn to meet in the Periphery of a Circle (or any other Figure) ſo as to be to each other in a given Ratio.

PROBLEM XIV.

From two given Points A, B, to draw two Right-lines AC, BC, which ſhall meet in a Right-line DE, given by Poſition, and alſo have a given Difference.

CONSTRUCTION.



Let PQ be the given Difference:

Draw AB, which biſect in F, and take FG a Third-proportional to 2 AB and PQ; alſo take GI = PQ, and draw GH and IK, perpendicular to AB, meeting DE in H and K; join H, A, and from the Center K, with the Interval AB, let an Arch be deſcribed cutting HA, produced (if need be), in L; draw LK, and AC, parallel to it, meeting DE in C; laſtly draw BC, *and the Thing is done.*

DEMONSTRATION.

Let CM be perpendicular to AB: Then, becauſe of the parallel Lines, KL (AB) : AC (:: KH : CH) :: GI (PQ) : GM, and conſequently AB × GM = AC × PQ (*by* 3. 4.) Moreover, by Conſtruction, 2 AB : PQ :: PQ : FG; therefore

fore

fore $AB \times FG = \frac{1}{2} PQ^2$; to which adding $AB \times GM = AC \times PQ$, we have $AB \times FM = AC \times PQ + \frac{1}{2} PQ^2$, and therefore $2\,AB \times FM = 2\,AC \times PQ + PQ^2$: But $2\,AB \times FM$ is alſo $= BC^2 - AC^2$ (*by Cor. to* 8. 2), therefore $2\,AC \times PQ + PQ^2$ is $= BC^2 - AC^2$, or $AC^2 + 2\,AC \times PQ + PQ^2 = BC^2$; that is $\overline{AC + PQ}^2 = BC^2$ (*by* 5. 2); whence $AC + PQ = BC$, and $BC - AC = PQ$. *Q. E. D.*

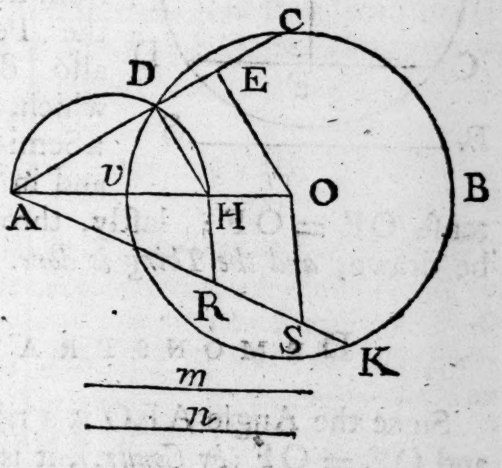
After the ſame Manner the Problem may be conſtructed, when the Sum, inſtead of the Difference, of the Lines is given.

PROBLEM XV.

From a given Point A, without a Circle BCv, given in Magnitude and Poſition, to draw a Right-line AC, in ſuch ſort that the Part thereof DC, inſcribed in the Circle, may be to the external Part AD in a given Ratio; viz. *that of n to m.*

CONSTRUCTION.

Draw AK at pleaſure, in which take $AR = m$, and $RS = \frac{1}{2} n$; alſo, from the Center O, draw OA; join O, S, and draw HR, parallel to OS, meeting AO in H; upon AH let a Semi-circle be deſcribed cutting the given Circle in D; laſtly draw ADC, *and the Thing is done.*



DEMONSTRATION.

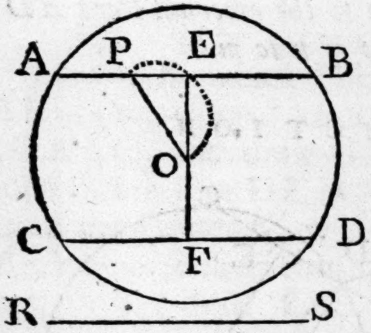
Upon AC let fall the Perpendicular OE, and join H, D.

The Angle ADH, in a Semi-circle, being a Right-angle (11. 3), DH is parallel to EO (9. 1), and conſequently $AD : DE\ (:: AH : HO) :: AR\ (m) : RS\ (\tfrac{1}{2}n)$, *by* 4. 4; whence, $AD : DC :: m : n$; becauſe $DC = 2\,DE$ (1. 3.). *Q. E. D.*

PROBLEM XVI.

Thro' a given Point P, in a given Circle ABDC, to draw a Right-line AB, which, terminating in the Periphery, ſhall be equal to a given Right-line RS.

CONSTRUCTION.



From any Point C in the Periphery, inſcribe the Subtenſe CD equal to RS (*by* 15. 5), upon which, from the Center O, let fall the Perpendicular OF; alſo draw OP, upon which, as a Diameter, let a Semi-circle be deſcribed, and in it apply the Subtenſe $OE = OF$; laſtly, thro' P and E let AB be drawn, *and the Thing is done.*

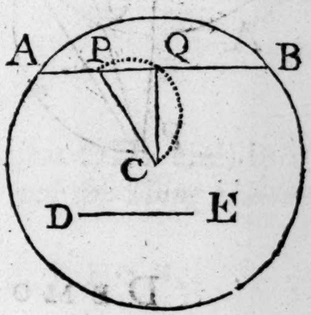
DEMONSTRATION.

Since the Angle AEO is a right one (*by* 11. 3), and $OE = OF$ (*by Conſtr.*), it is manifeſt that AB is $= CD$ (*by* 2. 3.) $= RS$. *Q. E. D.*

PROBLEM XVII.

Thro' a given Point P, in a given Circle C, to draw a Right-line AB; ſo that the Parts thereof AP, BP, intercepted by that Point and the Circumference of the Circle, may have a given Difference DE.

CONSTRUCTION.



From the Center C draw CP, upon which let a Semicircle PQC be deſcribed, and in it apply PQ equal to half DE, producing the ſame, both ways, to meet the Circumference in A and B, *and the Thing is done.*

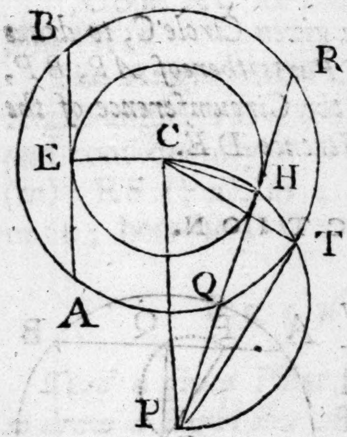
DEMONSTRATION.

Draw CQ, which will be perpendicular to AB (*by* 11. 3.), and therefore $AQ = BQ$, and ſo BP $(BQ + PQ) - AP\ (AQ - PQ) = 2PQ = DE$ (*by Conſtr.*). *Q. E. D.*

PROBLEM XVIII.

From a given Point P, without a Circle ARQ, given in Magnitude and Poſition, to draw a Right-line PR, to terminate in the oppoſite Periphery, ſo as to be divided, at its other Interſection with the Periphery, according to Extreme and Mean-proportion; or ſo that PR : QR :: QR : PQ.

CONSTRUCTION.



From the Center C draw CP; upon which let a Semicircle CTP be deſcribed cutting the given Circle in T; draw PT, and from any Point A inſcribe AB equal to it; make CE perpendicular to AB, and, upon the Center C, thro' E let another Circle EH be deſcribed interſecting CTP in H; draw PHR, *and the Thing is done.*

DEMONSTRATION.

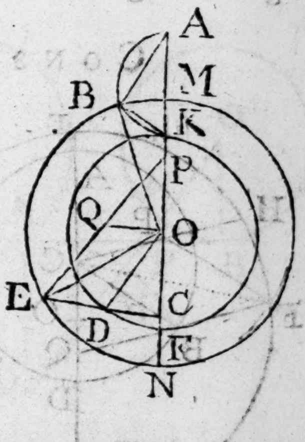
Join C, T and C, H.

The Angles PTC and PHC being both Right-angles (*by* 11. 3), it is evident that PT and PR are Tangents to the two concentric Circles ART and EH, and that QR (= AB, *by* 2. 3.) = PT: But $PR \times PQ = PT^2$ (QR^2) *by Cor. to* 17. 3; therefore PR : QR :: QR : PQ (*by* 3. 4). *Q. E. D.*

PROBLEM XIX.

To draw a Right-line EC, to make given Angles with a Right-line MN paſſing thro' the common Center O, of two given concentric Circles MEN, KDF, ſo that the Parts thereof CD, ED, intercepted by that Line and the Peripheries of the ſaid Circles, may obtain a given Ratio.

Construction.



Let QON be the given Angle which EC is to make with NC; and in OK, produced, let KA be ſo taken, as to be to the Radius OK, in the given Ratio of ED to CD; upon which let a Segment of a Circle ABK be deſcribed to contain an Angle equal to the Angle QOP; and, from its Interſection B, with the Circle MEN, draw BA, parallel to which, draw the Radius OD; and, thro' D, draw EC parallel to QO, *and the Thing is done.*

Demonstration.

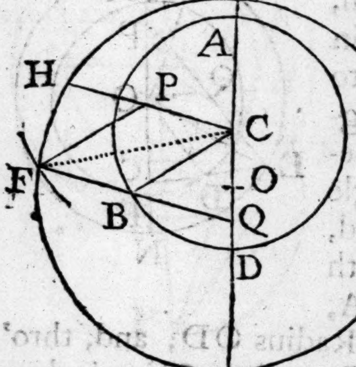
Draw BK, BO and EO; alſo draw EP, parallel to DO, meeting OQ and OM in Q and P.

Then, the Triangles AKB, POQ, having ABK = POQ (*by Conſtruction*), and KAB = OPQ (*by* 8. 1), have their external Angles OKB and EQO alſo equal to each other (*by* 10. 1.): Whence, becauſe EQ (= OD) = OK and EO = BO, therefore is OQ = KB (*by* 21. 1); therefore, the Triangles POQ, ABK being equiangular, PQ is alſo = AK (*by* 20. 1.): But (becauſe of the parallel Lines) QO (ED) : CD :: PQ (AK) : DO (OK). *Q. E. D.*

Problem XX.

To draw a Right-line FBQ, to make given Angles with a Right-line ER paſſing thro' the Centers of two *given Circles EFR, ABD, ſo that the Part thereof BF, intercepted by the two Peripheries, may alſo be of a given Length.*

CONSTRUCTION.



From the Center C, of the Circle ABD, let CH be drawn making the ſame Angles with ER, as FQ is to make; in which take CP equal to the given Length, and upon P, as a Center, with the Radius CA, let an Arch be deſcribed interſecting the Periphery EFR in F; draw FBQ parallel to HC, *and the Thing is done.*

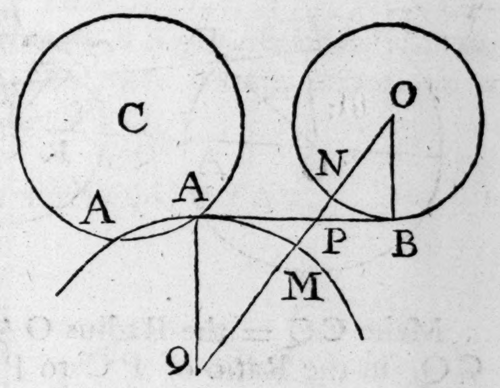
DEMONSTRATION.

Let PF, CF and CB be drawn: Then, becauſe the Triangles CBF, CPF have $CB = PF$ (*Conſtr.*), $CFB = FCP$ (8. 1.), and CF common; their other Sides BF and CP will alſo be equal (21. 1.). *Q. E. D.*

PROBLEM XXI.

To draw a Line AB, thro' a given Point P, to terminate in the Peripheries of two Circles C and O, given in Magnitude and Poſition, ſo that the Parts thereof PA, PB may obtain a given Ratio.

CONSTRUCTION.



From the Center O draw OP cutting the Circle O in N; which produce to Q, ſo that PQ may be to PO in the given Ratio of PA to PB; then take QM to ON in the ſame Ratio; and upon the Center Q, with the Interval QM, let a Circle be deſcribed interſecting the Circle C in the Point A; draw AQ, and alſo APB meeting OB, parallel to AQ, in B, *and the Thing is done.*

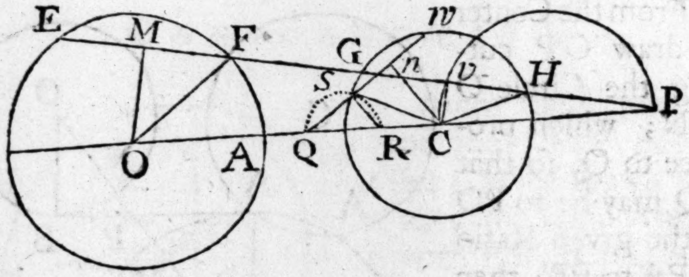
DEMONSTRATION.

Becauſe the Triangles QPA, OPB are equiangular (5. *and* 8. 1.), it is manifeſt that PQ : PO :: AP : BP (5. 4.). Moreover, becauſe QP : OP :: AQ (QM) : OB (*by* 5. 4.), and QP : OP : QM : ON (*by Conſtruction*), therefore is OB = ON; whence, it is, alſo, manifeſt, that the Point B falls in the Periphery of the Circle O. *Q. E. D.*

PROBLEM XXII.

From a given Point P, in a Right-line PCO paſſing thro' the Centers C and O of two Circles OEA, CwH, given in Magnitude and Poſition, ſo to draw a Right-line PE that the Parts thereof EF and GH, intercepted by thoſe Circles, may be equal.

CONSTRUCTION.



Make CQ = the Radius OA, and take CR to CQ, in the Ratio of PC to PO; upon QR and PC let two Semi-circles QSR and CvP be deſcribed, and thro' the Point S, where the former interſects the Circle CHG, draw Qw meeting the Periphery of the ſame Circle in w; alſo, in the other Semicircle inſcribe Cv = $\frac{1}{2}$ Sw, and thro' v draw PE, *and the Thing is done.*

DEMONSTRATION.

Draw OM and Cn perpendicular to PE and Qw reſpectively; alſo, let OF, RS, CS and CH be drawn.

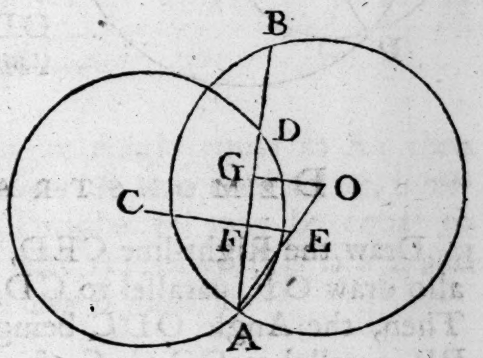
The Angles QSR, PvC being right ones (*by* 11. 3.), the Lines RS, Cn and Cv, OM will be parallel: Therefore, Sn : Qn ∷ CR : CQ (*by* 4. 4.) ∷ PC : PO (*by Conſtr.*); alſo Cv : OM ∷ PC : PO (*by* 5. 4.); whence (*by Equality*) Sn : Qn ∷ Cv : OM; therefore, Sn ($\frac{1}{2}$ Sw) being = Cv (*by Conſtruction*), Qn is alſo = OM; whence, as OF is = CQ (*by Conſtruction*), FM will be = Cn (*by.* 7. 1). But Cn is = Hv (*by* 7. 1.), becauſe Cv = Sn and CH = CS; therefore FM is alſo = Hv, and conſequently FE (2 FM) = HG (2 Hv.) *Q. E. D.*

PROBLEM XXIII.

From the Point of Interſection A, of two given Circles C and O, ſo to draw a Right-line AB, that the Subtenſes AB and AD, made thereby, may obtain a given Ratio.

CONSTRUCTION.

From the Center O draw OA, in which take AE to AO in the given Ratio of AD to AB (*by* 13. 5), and from the Center C draw CE, perpendicular to which draw AB, *and the Thing is done.*



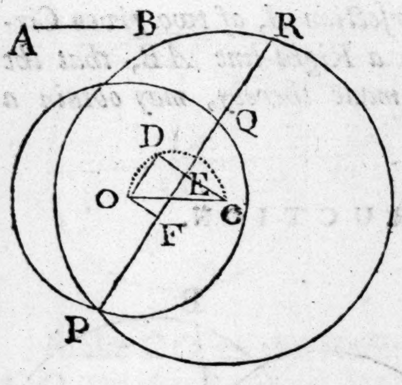
DEMONSTRATION.

Upon AB let fall the Perpendicular OG; then, becauſe of the Parallel-lines EF, OG, it will be, AE : AO :: AF : AG (5. 4.) :: AD (2 AF) : AB (2 AG). *Q. E. D.*

PROBLEM XXIV.

From the Point of Interſection P, of two given Circles O, C, ſo to draw a Right-line PR, that the Part thereof QR, intercepted by the two Peripheries, may be equal to a given Right-line AB.

CONSTRUCTION.



Upon the Right-line OC, joining the two Centers, let a Semicircle ODC be deſcribed, in which apply OD equal $\frac{1}{2}$ AB; and, parallel to OD, draw PR, *and the Thing is done.*

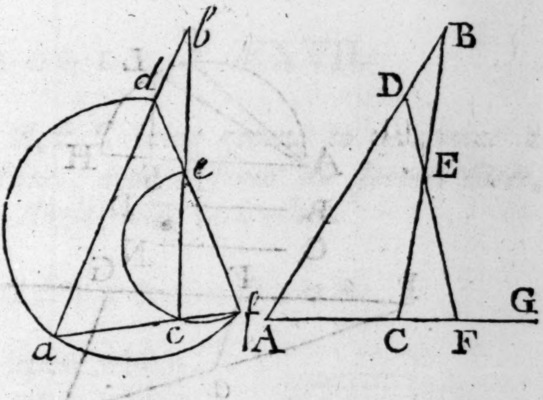
DEMONSTRATION.

Draw the Right-line CED, meeting PR in E, alſo draw OF, parallel to CD, meeting PR in F: Then, the Angle ODC being right (11. 3), and PR parallel to DO (*by Conſtruction*), E and F will alſo be Right-angles, and $EF = DO$ (23. 1.): Therefore, $PE - PF$ being $(= DO) = \frac{1}{2} AB$, it is manifeſt, that $2 PE\ (PR) - 2 PF\ (PQ) = AB$; or $QR = AB$. *Q. E. D.*

PROBLEM XXV.

To draw a Right-line DF to cut three Right-lines AB, AG and BC, given by Poſition, ſo that the Parts thereof DE, EF, intercepted by thoſe Lines, may be reſpectively equal to two given Right-lines de, ef.

CONSTRUCTION.



Upon ef deſcribe a Segment of a Circle fce capable of containing the Angle GCE (*by* 17. 5), and upon df ($de + ef$) let another Segment fad be deſcribed to contain an Angle equal to A; then draw fa (*by the precedent*) ſo that the Part ac, intercepted by the two Peripheries, may be equal to AC; join a, d, and take $AF = af$, $AD = ad$, and draw DF, *and the Thing is done.*

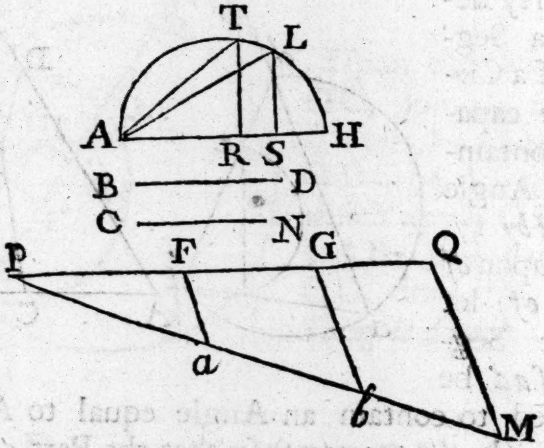
DEMONSTRATION.

The Triangles FAD, fad, having $AF = af$, $AD = ad$, and $A = a$ (*by Conſtruction*), are equal in all Reſpects; and therefore (if ce be drawn) the Triangles FCE, fce having $FCE = fce$, and $FC = fc$ (beſides $F = f$), will alſo be equal and alike: Therefore, ſeeing the Wholes DF, df, and the Parts FE, fe are equal, the remaining Parts DE and de muſt, conſequently, be equal. *Q. E. D.*

PROBLEM XXVI.

To divide a given Right-line PQ into three ſuch Parts, that their Squares may obtain the Ratio of three given Right-lines AH, BD and CN.

CONSTRUCTION.



Upon AH deſcribe a Semi-circle ATH; alſo in AH take $AS = BD$ and $AR = CN$; make SL and RT perpendicular to AH meeting the Periphery in L and T, and join A, L and A, T; from the Point P draw PM, making any Angle with PQ at pleaſure; in which take $Pa = AH$, $ab = AL$, and $bM = AT$; draw QM, parallel to which, draw *b*G and *a*F, *and the Thing is done.*

DEMONSTRATION.

Since $PF : FG :: Pa$ (AH) $: ab$ (AL, *by* 4. 4.), therefore is $PF^2 : FG^2 :: AH^2 : AL^2$ (*by Cor.* 1. *to* 9. 4.); but $AL^2 = AH \times AS$ (*by Cor. to* 11. 4.), conſequently $PF^2 : FG^2 :: AH^2 : AH \times AS :: AH : AS$ (BD, *by* 1. 4.). In the very ſame Manner it is proved that $PF^2 : GQ^2 :: AH : CN$. *Q. E. D.*

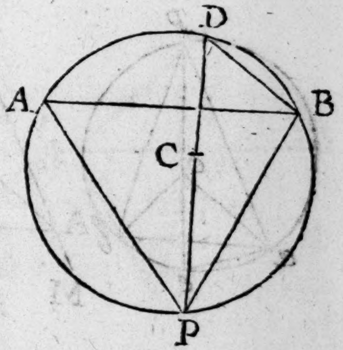
Note. By the ſame Method a given Right-line may be divided into any other Number of Parts, in ſuch ſort that ſimilar Figures made upon them (whether

(whether Squares, Triangles, &c.) may be to one another in any given Ratio.

PROBLEM XXVII.

Three Points A, B, C being given, to determine a fourth, where Lines, drawn from the former three, ſhall make given Angles with each other.

CONSTRUCTION.



Draw AB, upon which let a Segment of a Circle APB be deſcribed capable of containing the given Angle which the Lines drawn from A and B are to include, and let the whole Circle be completed; make the Angle ABD equal to the given Angle which the Lines drawn from A and C are to include; and from the Point D, where BD meets the Periphery, thro' C let DP be drawn, meeting the Periphery in P, *which will be the Point required.*

DEMONSTRATION.

Let AP and BP be drawn.

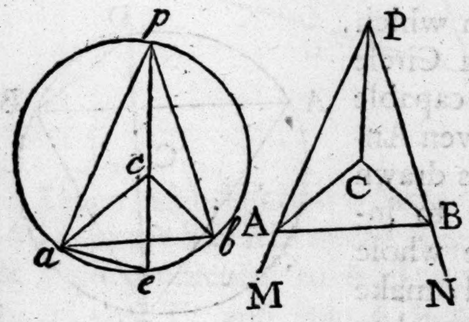
The Angle APB is equal to the given Angle, which the Lines drawn from A and B are to include (*by Conſtruction*). And, it is manifeſt that, APC = ABD, both ſtanding on the ſame Arch AD. *Q. E. D.*

PROBLEM

PROBLEM XXVIII.

From a given Point C to draw two Right-lines CA, CB to terminate in two other Right-lines PM, PN, given by Poſition, and which, together with the Line AB, joining their Extremes, ſhall form a Triangle ABC ſimilar to a given Triangle abc.

CONSTRUCTION.



Upon *ab* let a Segment of a Circle *apb* be deſcribed to contain an Angle equal to MPN, and let the whole Circle be completed; draw PC and alſo *ae* making the Angle *bae* = CPN, and interſecting the Periphery in *e*, and, thro' *c*, draw *ep* meeting the Periphery in *p*; make the Angles PCA, PCB reſpectively equal to the Angles *pca*, *pcb*; then join A, B, *and the Thing is done.*

DEMONSTRATION.

Let *pa* and *pb* be drawn.

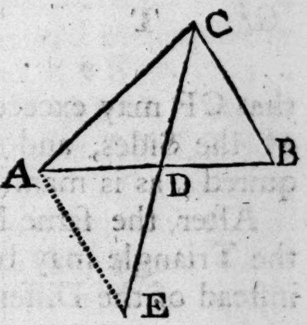
The Angle *bae* (CPB) is = *cpb*, both ſtanding on the ſame Arch *be*; therefore, APB being, alſo, = *apb* (*by Conſtruction*), the remaining Angles APC and *apc* muſt conſequently be equal; whence, as PCA = *pca*, and PCB = *pcb* (*by Conſtruction*), the Triangles APC, *apc* and BPC, *bpc* are equiangular, and ſo AC : *ac* (∷ PC : *pc*) ∷ CB : *cb*: Therefore the Triangles ABC, *abc*, having the

Sides about the equal Angles ACB, *acb* proportional, are like to each other (*by* 5. 4.). *Q. E. D.*

PROBLEM XXIX.

To deſcribe a Triangle, whereof the two Sides and the Right-line drawn from the vertical Angle to biſect the Baſe ſhall be given.

CONSTRUCTION.



Make CD equal to the given biſecting Line, in which, produced, take DE = CD, and from the Centers C and E, with Radii equal to the two given Sides, let two Arches be deſcribed interſecting each other in A; draw ADB, and make DB = AD; draw CA and CB, and then ABC will be the Triangle required.

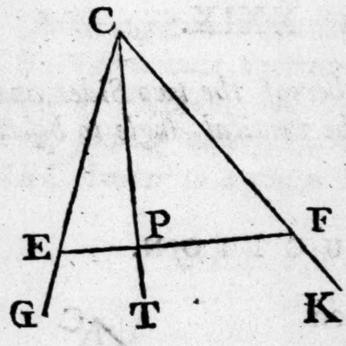
DEMONSTRATION.

Let AE be drawn: Then, becauſe CD = DE, DB = DA (*by Conſtruction*), and alſo the Angle CDB = the Angle EDA, therefore will CB be likewiſe equal to AE. *Q. E. D.*

PROBLEM XXX.

To deſcribe a Triangle, whereof the vertical Angle, and the Difference of the Sides including it, ſhall be given; and wherein a Right-line drawn from the Vertex, making given Angles with thoſe Sides and terminating in the Baſe, ſhall alſo be of a given Length.

CONSTRUCTION.



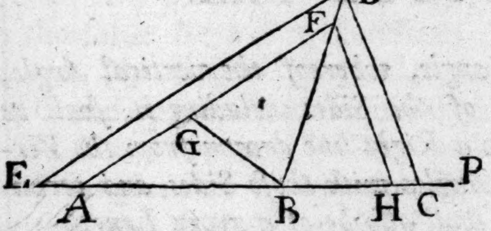
Make GCK equal to the given vertical Angle, and draw CT ſo as to make the Angles GCT and KCT equal to the other two given Angles; in CT take CP equal to the given Line; then thro' P, *by Prob.* 10*th*, let the Right-line EPF be drawn ſo that CF may exceed CE by the given Difference of the Sides, and CEF will be the Triangle required; as is manifeſt from the Conſtruction.

After the ſame Manner, by help of Prob. 9th, the Triangle may be conſtructed, when the Sum, inſtead of the Difference, of the two Sides is given.

PROBLEM XXXI.

To deſcribe a Triangle, whereof the Difference of the Sides, and the Difference of the Segments of the Baſe (made by a Perpendicular falling from the Vertex) ſhall be given, and whereof the Difference of the Angles at the Baſe ſhall be equal to half the Angle at the Vertex.

CONSTRUCTION.



In the indefinite Right-line EP, take BC equal to the Difference of the Sides, and BA equal to the Difference of the Segments of the Baſe; and upon BC make an iſoceles ſceles Triangle BCD, whoſe Side BD or CD is equal to BA; alſo make the Angle BDE = BDC; draw AF, parallel to ED, meeting BD in F, and FH, parallel to DC, meeting EP in H; then will AHF *be the Triangle required.*

DEMONSTRATION.

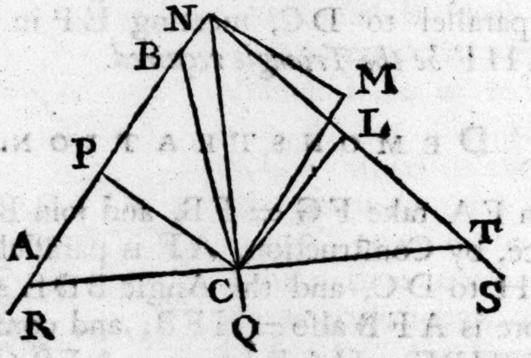
In FA take FG = FB, and join B, G.

Since, by Conſtruction, AF is parallel to ED, and FH to DC, and the Angle BDE = CDB, therefore is AFB alſo = HFB; and conſequently AHF (HBF)—HAF, being = AFB (*by* 10. 1.), is equal to half the vertical Angle AFH. Moreover, becauſe the iſoſceles Triangles BFG, BFH have BFG = BFH and one Side BF common, not only their Baſes BG, BH, and remaining Angles BGF, HBF, *&c.* but alſo AGB and ABF the Supplements of thoſe Angles will be equal: Therefore, the Triangles AGB, ABF, as well as BFH and BDC being equiangular, it will be, BG : AG :: BF : AB (BD) :: BH : BC (*by* 5. 4.); whence, becauſe the Antecedents BG, BH are equal, the Conſequents AG and BC are alſo equal, and therefore AF — FH (AG) = BC. *Q. E. D.*

PROBLEM XXXII.

To deſcribe a Triangle, whereof one Angle, the Sum of the two Sides including it, and the Perpendicular, falling from thence on the oppoſite Side, ſhall be all given.

CONSTRUCTION.



Make RNQ and SNQ each equal to half the given Angle, and make NM perpendicular to NR, and equal to the given Perpendicular; moreover draw MC, parallel to NR, meeting NQ in C, and thro' the Point C let AT be drawn, *by Prob. VII.* equal to the given Sum of the Sides, ſo as to terminate in NR and NS; make the Angle ACB equal to ANT, *and ACB will be the Triangle required.*

DEMONSTRATION.

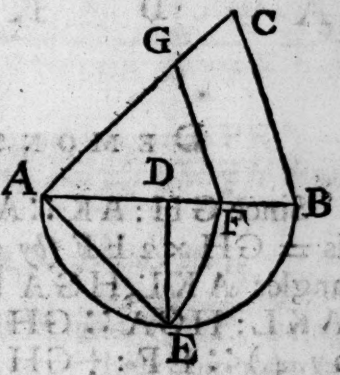
Upon NR and NS let fall the Perpendiculars CP and CL.

The Angle ACB is equal to the given Angle, *by Conſtruction*; and, it is manifeſt, that PC is alſo equal to (NM) the given Perpendicular. Moreover, the Triangles ACB, ATN, having ACB = ANT (*by Conſtr.*), and A common, have alſo ABC (PBC) = ATN (CTL); therefore, ſince CP = CL (becauſe CNP = CNL, *by Conſtr.*), it is likewiſe evident, that CB = CT, and conſequently AC+CB (=AC+CT) = AT. *Q. E. D.*

PROBLEM XXXIII.

To draw a Right-line FG parallel to one Side BC of a given Triangle ABC, ſo that the Triangle AFG, cut off thereby, may be to the whole ABC in a given Ratio.

CONSTRUCTION.



Let the given Ratio be that of AB to AD; upon AB let a Semi-circle AEB be deſcribed; and draw DE, perpendicular to AB, meeting the Periphery in E; thro' E, from the Center A, deſcribe EF interſecting AB in F; draw FG parallel to BC, *and the Thing is done.*

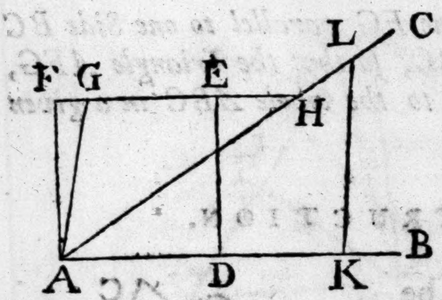
DEMONSTRATION.

Let AE be drawn; then AE^2 (AF^2) $= AD \times AB$ (*by Cor. to* 11.4.): But $ABC : AFG :: AB^2 : AF^2$ $(AD \times AB)$ $:: AB : AD$ (*by* 17. *and* 1. 4.). *Q. E. D.*

PROBLEM XXXIV.

To draw a Right-line KL, parallel to a given Right-line AG, which ſhall cut two other Right-lines AC, AB, given by Poſition, in ſuch ſort that the Triangle AKL, formed thereby and the two given Lines, may be equal to a given Rectangle ADEF.

CONSTRUCTION.



Let FE produced (if need be) meet AG and AC in G and H; and, in AB, take AK a Mean-proportional between GH and 2 EF; draw KL parallel to AG, *and the Thing is done.*

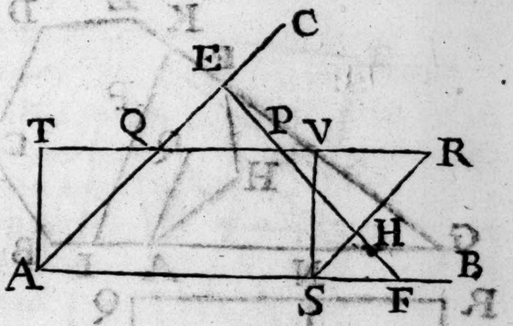
DEMONSTRATION.

Since $GH : AK :: AK : 2EF$ (*by Conſtr.*), AK^2 is $= GH \times 2EF$ (*by* 3.4.): Therefore, the Triangles AKL, HGA being equiangular, we have $AKL : HGA :: GH \times 2EF\ (AK^2) : GH^2$ (*by* 17.4.) $:: EF : \frac{1}{2} GH :: EF \times AF$ (ADEF) $: \frac{1}{2} GH \times AF$ (HGA, *by* 1.4.): Therefore, the Conſequents being equal, the Antecedents AKL and ADEF muſt alſo be equal. *Q. E. D.*

PROBLEM XXXV.

To draw a Right-line EF, thro' a given Point P, between two Right-lines AB, AC, given by Poſition, ſo that the Triangle AEF, formed thereby and the two given Lines, may be equal to a given Rectangle ST.

CONSTRUCTION.



Thro' P, parallel to A B, let QPR be drawn meeting AC in Q; and upon AQ let a Parallelogram ASRQ be deſcribed equal to ST (*by* 7. 6); in SB take SF equal to the Side of the Square which is $= PR^2 - PQ^2$ (*by* 3. 6.); draw FPE, *and the Thing is done.*

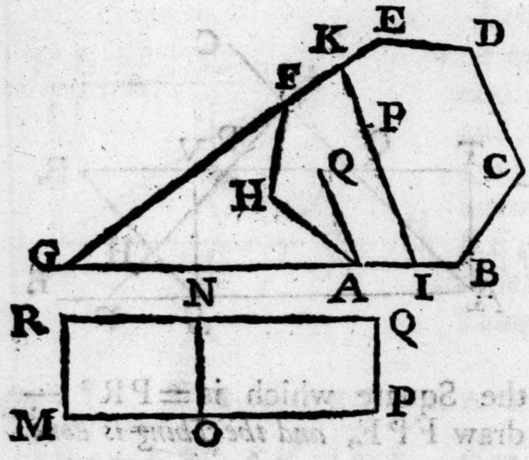
DEMONSTRATION.

The Triangles SFH, PRH and QPE, being equiangular, are as the Squares of their homologous Sides (*by* 17. 4.); therefore, ſince $SF^2 + PQ^2 = PR^2$ (*by Conſtruction*), it is manifeſt, that SFH + QPE is alſo = PRH; to which let ASHPQ, common, be added, and then AFE = ASRQ (ST). *Q. E. D.*

PROBLEM XXXVI.

From a given Polygon ABCDEFH, to cut off a Part AHFKI equal to a given Rectangle MN, by a Right-line IK, either, parallel to a given Right-line AQ, or paſſing thro' a given Point P.

CONSTRUCTION.



Let BA and EF be produced to meet each other in G; and upon ON let a Rectangle OQ be conſtituted *(by* 7. 6.) equal to AGFH; then, by one of the two laſt Problems (according to the Caſe propoſed), draw IK ſo as to make the Triangle GKI $=$ MQ, *and the Thing is done.*

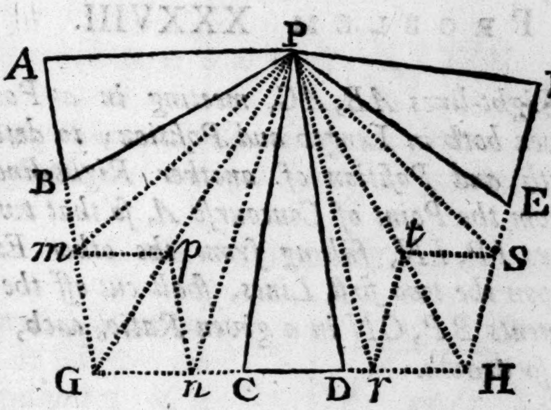
The Demonſtration whereof is manifeſt from the Conſtruction.

After the ſame manner the Polygon may be divided according to any given Ratio; becauſe, the whole being given, each Part will alſo be given.

PROBLEM XXXVII.

To determine the Poſition of a Point P, from whence Lines drawn to the Extremes of three Right-lines AB, CD, EF, given in Length and Poſition, may form three Triangles ABP, CPD, EPF mutually equal to each other.

CONSTRUCTION.



Let the given Lines, produced, meet each other in G and H; in which take Gm = AB, H*s* = EF, and Gn, Hr, each, equal to CD; compleat the Parallelograms Gmpn, Hrts, and let the Diagonals Gp, Ht be produced till they meet in P, *and the Thing is done.*

DEMONSTRATION,

Let PA, PB, Pm, *&c.* be drawn.

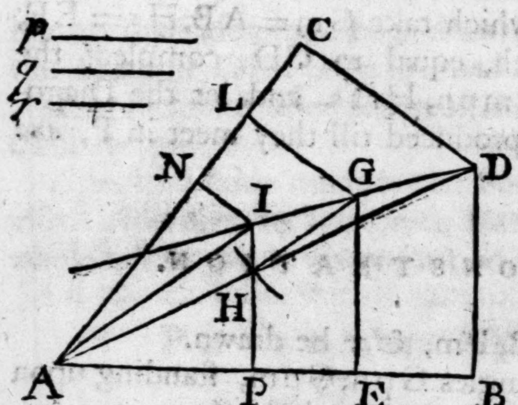
Since the Triangles Gpn, Gpm, ſtanding upon the ſame Baſe Gp, are equal (*by Cor.* 1. *to* 2. 2), the Triangles GPn, GPm will alſo be equal (*by Cor.* 2. *to* 2. 2); but CPD = GPn, and APB = GPm (*by Cor.* 2. *to* 2. 2.); whence CPD and APB are likewiſe equal: By the very ſame Reaſoning, it is proved, that CPD and FPE are equal. *Q. E. D.*

Note. This Problem may be conſtructed, when the Triangles, inſtead of being equal, are in a given Ratio; by taking Gn to CD in the Ratio of APB to CPD; and Hr to CD in the Ratio of E P F to C P D ; all the reſt of the Conſtruction being exactly the ſame.

PROBLEM XXXVIII.

Two Right-lines AB, AC, meeting in a Point A, being given both in Length and Poſition; to determine the Length and Poſition of another Right-line AI, drawn from the Point of Concourſe A, ſo that two Perpendiculars IP, IN, falling from the other Extreme thereof upon the two firſt Lines, ſhall cut off the alternate Segments BP, CN in a given Ratio, each, to AI the Line ſo drawn.

CONSTRUCTION.



Let the Lines AI, BP, and CN be to each other as p, q, and r reſpectively. Take BE = q, CL = r, and make BD, EG, CD, and LG perpendicular to AB and AC, meeting in D and G; draw DA and DG, and from the Center G, with the Interval *p*, let an Arch be deſcribed interſecting AD in H; draw HG, and AI, parallel to it, meeting DG produced (if need be) in I, *and the Thing is done.*

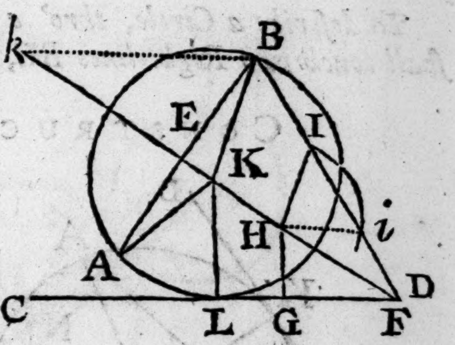
DEMONSTRATION.

Becauſe of the parallel Lines, it will be AI : GH (:: ID : GD) :: BP : BE ; whence, alternately, ly, A I : BP :: G H (*p*) : BE (*q*). By the ſame Method of Arguing it will appear that A I : CN :: *p* : *r*. *Q. E. D.*

Problem XXXIX.

To deſcribe a Circle, thro' two given Points A, B, which ſhall touch a Right-line C D *given by Poſition.*

Construction.



Draw A B, which biſect in E, by the Perpendicular E F meeting C D in F; alſo from any Point H in that Line draw HG perpendicular to CD; from the Center H, with the Interval HG, let an Arch be deſcribed cutting BF (the Line joining B, F) in I; draw I H, and B K parallel to it meeting E F in K, and from the Center K, with the Radius B K, let a Circle be deſcribed, *and the Thing is done.*

Demonstration.

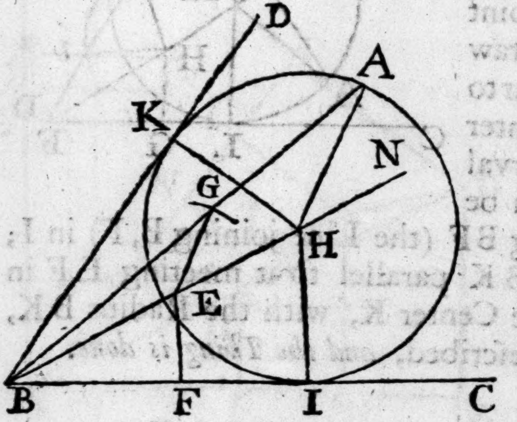
Join K A and draw K L perpendicular to C D: Then, becauſe of the parallel Lines K L, H G; K B, H I, it will be H G : H I :: K L : K B; whence as H G and H I are equal, *by Conſtruction*, K L and K B are equal likewiſe. But it is evident, from the Conſtruction, that K A is = K B; therefore K B = K L = K A. *Q. E. D.*

Note. Since the Arch, which cuts B F in I, will alſo (if continued) cut it in another Point *i*, it is evident, from the ſame Demonſtration, that, if B *k* be drawn, parallel to H *i*, interſecting F E in *k*, the Point *k* will be the Center of another Circle anſwering the Conditions required. Underſtand the like in the Conſtruction of the two ſucceeding Problems.

PROBLEM XL.

To deſcribe a Circle, thro' a given Point A, which ſhall touch two Right-lines B C, B D given in Poſition.

CONSTRUCTION.



From B the Point of Concourſe of the two given Lines draw B A, and biſect the Angle D B C by the Right-line B N, from any Point E in which, upon B C, let fall the Perpendicular E F; alſo from the ſame Point E, as a Center, with the Radius E F, let an Arch be deſcribed interſecting B A in G; draw G E, and A H, parallel to it, meeting B E in H; laſtly, from the Center H, with the Interval H A, let a Circle be deſcribed, *and the Thing is done.*

DEMONSTRATION.

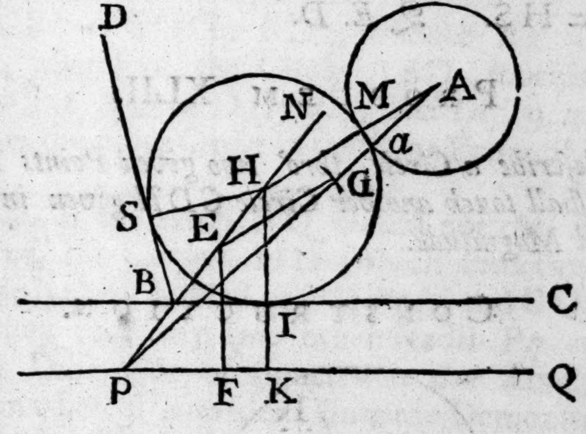
Upon B C and B D let fall the Perpendiculars H I and H K, which are, manifeſtly, equal (*from the*

Conſtruction): Moreover, by reaſon of the parallel Lines, we have EF : EG :: HI : HA; whence, EF and EG being equal, HI and HA are alſo equal. *Q. E. D.*

PROBLEM XLI.

To deſcribe a Circle which ſhall touch two Right-lines BC and BD, given by Poſition, and likewiſe another Circle AaM, given in Magnitude and Poſition.

CONSTRUCTION.



Draw PQ parallel to BC, at the Diſtance of the Radius Aa; and, thro' B, the Point of Concourſe of the two given Lines, draw the Right-line NBP, biſecting the Angle CBD, and meeting PQ in P; moreover, from any Point E in PN, upon PQ, let fall the Perpendicular EF, and from the ſame Point, as a Center, with the Radius EF, let an Arch be deſcribed interſecting the Line joining P, A in G; draw GE, and AH, parallel thereto, meeting the Periphery of the given Circle in M and the Right-line PN in H, from which laſt

Point,

Point, as a Center, thro' M, deſcribe a Circle, *and the Thing is done.*

DEMONSTRATION.

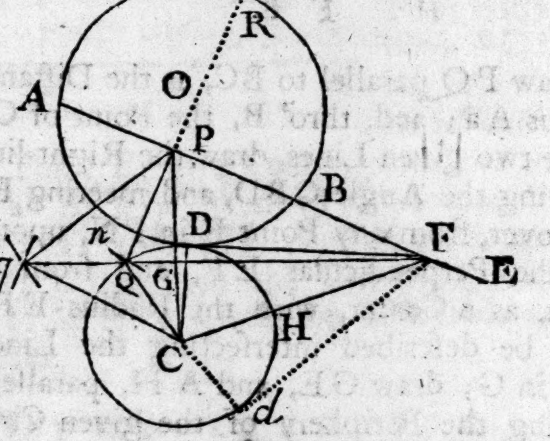
Draw HS perpendicular to BD, and HK to PQ, interſecting BC in I.

Becauſe of the parallel Lines, we have EG : EF :: HA : HK; therefore, EG and EF being equal (*by Conſtr.*), HA and HK are likewiſe equal; from which take away the equal Quantities KI and AM (A*a*), and the Remainders HI and HM will be equal: But, it is evident, that HI is = HS, becauſe BH biſects the Angle IBS; therefore HI = HM = HS. *Q. E. D.*

PROBLEM XLII.

To deſcribe a Circle, thro' two given Points A, B, which ſhall touch another Circle CDH given in Poſition and Magnitude.

CONSTRUCTION.



Through A and B, the given Points, draw the indefinite Line AE, and from C, the Center of the

the given Circle, draw CP biſecting AB in P, upon which Point, as a Center, at the Diſtance of PA (or PB), let an Arch be deſcribed interſecting the Periphery HDQ in Q; from Q, perpendicular to CP, draw QGF, meeting AE in F; and from F draw FD to touch the Circle CH in D; then thro' the three Points A, B, D let a Circle be deſcribed, *and the Thing is done.*

DEMONSTRATION.

Let CF, CD, CQ and PQ be drawn: Then, $PF^2 - PQ^2 (PA^2) = (FG^2 - QG^2) CF^2 - CQ^2 (CD^2)$; but $PF^2 - PA^2 = AF \times BF$ (*by* 17. 3.), and $CF^2 - CD^2 = FD^2$; whence $AF \times BF = FD^2$; therefore the Circle ABD, touching the Right-line AD in D (*by* 5. 3, *and Def.* 9. 3.), alſo touches the other Circle in the ſame Point. *Q. E. D.*

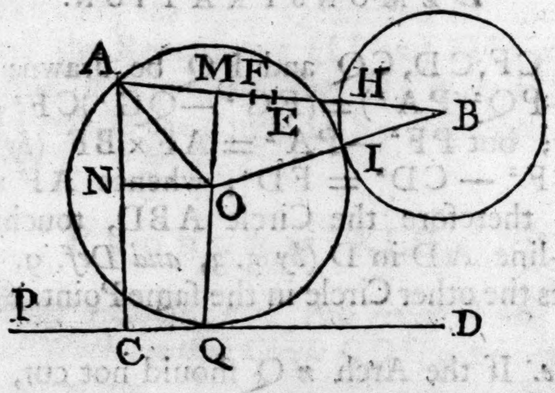
Note. If the Arch *n* Q ſhould not cut, but fall ſhort of, the Circle CHD (which ſometimes will happen); then, inſtead of PQ (PA) and CQ (CD), you may take any two other Radii P*q* and C*q* whoſe Difference of Squares is the ſame. The Reaſon whereof is evident from the Demonſtration; which, in this Caſe, will be exactly the ſame as above. Moreover, if you would have the required Circle be touched inwardly by the given Circle, the Proceſs will be the very ſame; only, inſtead of FD, you muſt take the other Tangent F*d* drawn from the ſame Point.

The ſeveral Caſes of this Problem may likewiſe be conſtructed by help of *Prob. 14th*; that is, by finding a Point O, in the Right-line PR, perpendicular to the Middle of AB, from whence Lines, drawn to the given Points B and C, may have the given Difference CD; the Point ſo found being, manifeſtly, the Center of the required Circle.

PROBLEM XLIII.

To deſcribe a Circle, thro' a given Point A, which ſhall touch a Right-line P D, given by Poſition, and likewiſe another Circle B H I, given in Magnitude and Poſition.

CONSTRUCTION.



Upon PD let fall the Perpendicular AC, and from the Center B draw BA, cutting the Periphery of the given Circle in H; biſect AB in E, and take EF a Third-proportional to 2 AB and BH; and draw AO (*by Prob.* 38.) ſo, that, the Perpendiculars ON and OM being let fall upon AC and AB, the Lines AO and CN may be equal one to the other, and have each the Ratio to FM which AB has to BH: Then, from the Center O, thro' A, let a Circle be deſcribed, *and the Thing is done.*

DEMONSTRATION.

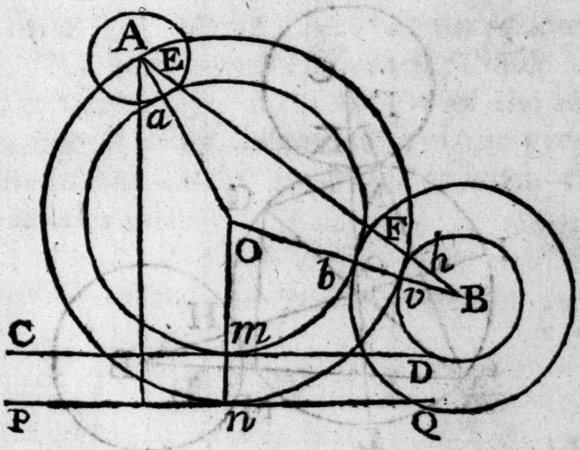
Since (*by Conſtr.*) $CN = AO$, it is plain that the Circle O touches the Right-line PD: But, to prove that it alſo touches the Circle B, let BO be drawn, interſecting the Circle OAQ in the Point I; then, ſince (*by Conſtr.*) $2AB : BH :: BH : EF$, and

$AB : BH :: AO : FM$, we ſhall have $AB \times EF = \frac{1}{2}BH^2$, and $AB \times FM = BH \times AO$; whence $AB \times EM$ $(AB \times EF + AB \times FM) = \frac{1}{2}BH^2 + BH \times AO$, and conſequently $AB \times 2EM = BH^2 + 2AO \times BH$; but $AB \times 2EM$ is $= BO^2 - AO^2$ (*by* 8. 2.), therefore $BO^2 - AO^2$ is likewiſe $= BH^2 + 2AO \times BH$, and conſequently $BO^2 = BH^2 + 2AO \times BH + AO^2 = \overline{BH + AO}|^2$ (*by* 5. 2.); whence, alſo, $BO = BH + AO$, and therefore, by taking away $IO = AO$, we have $BI = BH$; whence, it is manifeſt, that the Circle O touches the Circle B in I. *Q. E. D.*

PROBLEM XLIV.

To deſcribe a Circle which ſhall touch a Right-line CD, given in Poſition, and two other Circles AEa and BFb, given in Magnitude and Poſition.

CONSTRUCTION.



From the Radius BF, of the greater Circle, take away F*b* equal to the Radius AE of the leſſer; and from the Center B, with the Interval B*b*, deſcribe

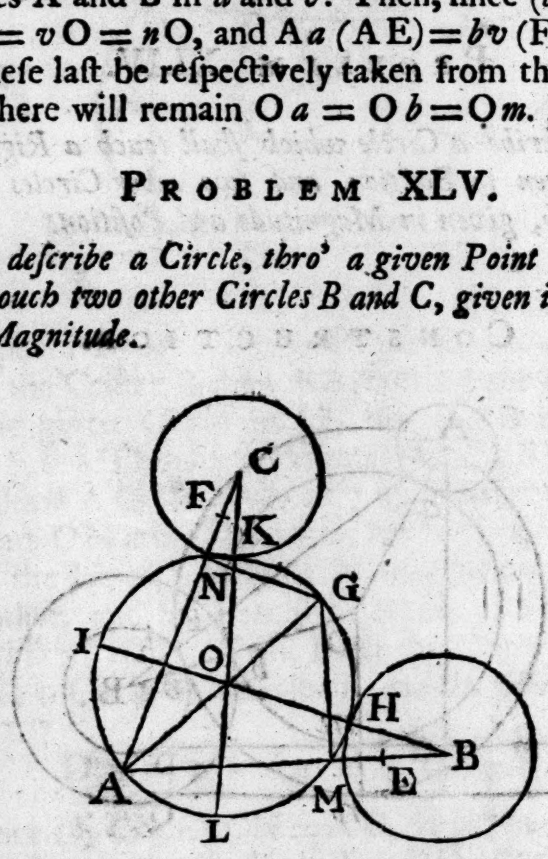
ſcribe the Circle B*b v*; alſo, draw PQ, parallel to CD, at the Diſtance of F*b* or AE: Then, by the laſt Problem, let the Center O of a Circle, paſſing thro' A and touching PQ and B*b v*, be found, and the ſame Point O will, likewiſe, be the Center of the required Circle O*amb*.

DEMONSTRATION.

Draw O*n*, perpendicular to PQ, cutting CD in *m*; alſo let OA and OB be drawn interſecting the Circles A and B in *a* and *v*: Then, ſince (*by Conſtr.*) AO = *v*O = *n*O, and A*a* (AE) = *bv* (F*b*) = *nm*, let theſe laſt be reſpectively taken from the former, and there will remain O*a* = O*b* = O*m*. *Q. E. D.*

PROBLEM XLV.

To deſcribe a Circle, thro' a given Point A, which ſhall touch two other Circles B and C, given in Poſition and Magnitude.



Suppoſe the Thing done; and thro' the Center O, of the required Circle, ſuppoſe IB and AG to be be drawn, meeting the Periphery thereof in I, H, and G; moreover, from the Points M, N where the Lines AB and AC (joining the given Point and Centers) interſect that Periphery, conceive MG and NG to be drawn, which will be perpendicular to AB and AC (*by* 11. 3.). Furthermore, ſince $AB \times BM$ is $= BI \times BH$ (*by* 17. 3.) $= \overline{AG + BH} \times BH = AG \times BH + BH^2$, we ſhall have $AB \times BM - BH^2 = AG \times BH$; which, if BE be taken a Third-proportional to AB and BH, or $AB \times BE$ be made $= BH^2$, will become $AB \times EM$ $(AB \times BM - BH^2) = AG \times BH$; and therefore $AG : EM :: AB : BH$. By the ſame Argument, if CF be taken a Third-proportional to AC and CK, it will appear that $AG : FN :: AC : CK$. Wherefore, ſeeing both AE and AF are actually given, and that EM and FN have, each, a given Ratio to AG, the Method of Conſtruction is manifeſt, from *Prob. XXXVIII.*

After the ſame Manner a Circle may be deſcribed to touch three given Circles; the Problem, in either Caſe, amounting to no more than, *To find a Point from whence Lines, drawn to three given Points, ſhall have given Differences*: Since a Point ſo found will always be the Center of the required Circle, as well when one or more of the given Circles touch that Circle inwardly, as when they all touch it outwardly.

FINIS.

MVSEVM BRITANNICVM

ERRATA.

PAGE 18, line 18, for *eqaal*, read *equal*; In the Quotation *b*, to *Theor*. 12, f. 10. 1, r. *Cor*. 1. *to* 10. 1.; p. 19, l. 28, f. *iſoſeles*, r. *iſoſceles*; p. 20. after the Quotation *c* to *Theor*. 16, r. the Quotation *d Cor*. 6. *to* 10. 1; p. 26. in the Quotation *c* to *Theor*. 23, f. 10. 1, r. *Cor*. 2. *to* 10. 1, ; and in the Quotation *b* to *Theor*. 24, f. 18, r. 19; p. 29, in the Quotation *a*, f. 26, r. 27; p. 32, in the Quotation *c*, f. 10. 1, r. *Cor*. 2. *to* 10. 1; p. 64, l. 27, f. *ſhonld*, r. *ſhould*; p. 66, l. 27, after equal, f. *e*, r. *c*; and l. 29, after BED, f. *c*, r. *e*; p. 68, l. laſt, after D, f. *a*, r. *b*; p. 70. l. 19, after G c, r. *and*; p. 72, l. laſt, before A D, r. *and*; p. 74, l. 2, f. *Rigt*, r. *Right*; l. 27, f. *terminating* r. *cutting*; p. 78, l. 6, f. *CD*, r. CD^2; p. 81, l. 6, after *Angle*, r. in.